Stability of sorting based embeddings

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Abstract

Consider a group G of order M acting unitarily on a real inner product space V. We show that the sorting based embedding obtained by applying a general linear map $\alpha : \mathbb{R}^{M\times N} \to \mathbb{R}^D$ to the invariant map $\beta_\Phi : V \to \mathbb{R}^{M\times N}$ given by sorting the coorbits $(\langle v, g\phi_i\rangle_V)_{g\in G}$, where $(\phi_i)_{i=1}^N \in V$, satisfies a bi-Lipschitz condition if and only if it separates orbits.

Additionally, we note that any invariant Lipschitz continuous map (into a Hilbert space) factors through the sorting based embedding, and that any invariant continuous map (into a locally convex space) factors through the sorting based embedding as well.

1 Introduction

Suppose that a specific learning task with input space V is invariant under the action of a group G. Then, it makes sense to construct a class of hypotheses that factor into two parts: i. a group invariant part $h : V \to V_{\text{int}}$, where V_{int} is some intermediate space, and ii. an "optimisable" part $g : V_{\text{int}} \to W$, where W denotes the space of all possible outputs. In this way, we can use an optimisation algorithm to find g in such a way that $f = g \circ h$ fits some training data, and the above construction ascertains that $f : V \to W$ is invariant under the action of G.

So, introducing the equivalence relation $v \sim w :\iff \exists g \in G : v = gw$ on V and assuming that V has a norm $\|\cdot\|_V$, our goal becomes to construct maps $h : V \to \mathbb{R}^D$, where $D \in \mathbb{N}$ is as small as possible, and h satisfies the properties:

1. *Invariance.* $h(v) = h(w)$ for all $v, w \in V$ such that $v \sim w$.
2. *Orbit separation.* $v \sim w$ for all $v, w \in V$ such that $h(v) = h(w)$.
3. *Bi-Lipschitz condition.* There exist constants $0 < c \leq C$ such that

$$c\,\mathrm{dist}(v, w) \leq \|h(v) - h(w)\|_2 \leq C\,\mathrm{dist}(v, w), \qquad v, w \in V. \tag{1}$$

Here and throughout the paper, $\mathrm{dist}(v, w) := \min_{g\in G}\|v - gw\|_V$ denotes the natural metric on the quotient space $V/\!\sim$.

The approach described above is an instance of *invariant machine learning*: techniques designed to ensure that hypotheses are robust to specific changes in the input data. More precisely, it is a special form of *feature engineering*, where raw data is transformed into a more useful set of inputs.

Feature transforms have been suggested in computer vision, for example, where handcrafted maps such as the scale-invariant feature transform (SIFT) [18] or the histogram of oriented gradients (HOG) [14] were created to be invariant to transformations like scaling, translation or rotation. The results we present here were inspired by the two more recent papers [12, 16] and are a continuation of work presented in [3, 4, 5]. Similar approaches can also be found in [1, 10, 11] and, most notably, [19].

Taking a wider view of the literature, there is another notable approach to invariant machine learning which consists of propagating the invariance of a problem through multiple *equivariant layers*: maps $f : V \to W$ such that $f(gv) = gf(v)$ for all $g \in G$ and all $v \in V$. (Here, G is also assumed to act on W.) The posterchildren for equivariant machine learning models are convolutional neural networks (CNNs) [17] which are translation invariant. (Though some care has to be taken when defining "translation invariance" [2].)

A well-known generalisation of the CNN architecture in the same spirit is the group equivariant convolutional network architecture [13] which introduces layers that can respect more general symmetries. Alternative approaches include [22] (cf. also [9]) as well as [23].

1.1 Sorting-based embeddings

Let G be a finite group acting unitarily on a $d \in \mathbb{N}$ dimensional real inner product space V. One approach to construct maps that satisfy items 1 through 3 is to enumerate the group $G = \{g_i\}_{i=1}^M$ and define the coorbits

$$\kappa_\phi : V \to \mathbb{R}^M, \qquad \kappa_\phi v := \left(\langle v, g_1\phi\rangle_V \quad \ldots \quad \langle v, g_M\phi\rangle_V\right)^\top,$$

for $\phi \in V$, where $\langle\cdot,\cdot\rangle_V$ denotes the inner product on V. By choosing a finite sequence $\Phi := (\phi_i)_{i=1}^N \in V$, sorting the coorbits, and collecting them in a matrix, one obtains an invariant map

$$\beta_\Phi : V \to \mathbb{R}^{M\times N}, \qquad \beta_\Phi(v) := \left(\mathrm{sort}(\kappa_{\phi_1} v) \quad \ldots \quad \mathrm{sort}(\kappa_{\phi_N} v)\right),$$

where $\mathrm{sort} : \mathbb{R}^M \to \mathbb{R}^M$ denotes the operator that sorts vectors in a monotonically decreasing way. To reduce the embedding dimension, a linear map $\alpha : \mathbb{R}^{M\times N} \to \mathbb{R}^D$ can be applied to the sorted coorbits.

This idea was first introduced in [3] in the context of the action of the group S_m by row permutation on the space of matrices $\mathbb{R}^{m\times n}$. The authors demonstrated that the map $\gamma(\mathbf{X}) := \mathrm{sort}(\mathbf{XA})$[1] separates orbits for full spark matrices $\mathbf{A} \in \mathbb{R}^{n\times N}$ with $N > (n-1)m!$. They also showed that γ satisfies the bi-Lipschitz condition (inequality (1)) if it separates orbits, and that $\alpha' \circ \gamma$ satisfies inequality (1) for a generic[2] linear map $\alpha' : \mathbb{R}^{m\times N} \to \mathbb{R}^D$ with $N \geq 2n$ and $D \geq 2mn$ (provided that γ separates orbits). Notably, the embedding $\alpha' \circ \gamma$ passes through an intermediate Euclidean space of dimension larger than $(n-1)m!$, which grows rapidly with the matrix size.

The papers [12, 16] address this issue. In [16], among other things, it is shown that $\gamma(\mathbf{X}) := \mathrm{diag}(\mathbf{B}^\top \mathrm{sort}(\mathbf{XA}))$ separates orbits for generic pairs $(\mathbf{A}, \mathbf{B}) \in \mathbb{R}^{n\times D} \times \mathbb{R}^{D\times m}$ with $D > 2mn$. Whether γ satisfies inequality (1) remained

[1] Here, $\mathrm{sort} : \mathbb{R}^{m\times N} \to \mathbb{R}^{m\times N}$ is the operator that sorts matrices column-wise.

[2] By *generic*, we mean that a statement is true in a non-empty Zariski open set.

an open question. In [12], the authors show that, if $\alpha(\mathbf{X}) := \begin{pmatrix} X_{11} & \dots & X_{1D} \end{pmatrix}$ selects the maximal entries of the coorbits, then $\gamma := \alpha \circ \beta_\Phi : \mathbb{R}^d \to \mathbb{R}^D$ separates orbits for generic sequences $\Phi = (\phi_i)_{i=1}^D \in \mathbb{R}^d$ with $D \geq 2d$, whenever $G \leq O(d)$ is a finite subgroup of the orthogonal $d \times d$ matrices. They also prove that γ satisfies inequality (1) with high probability if D is sufficiently large, and they pose the question whether γ satisfies inequality (1) whenever it separates orbits.

This was affirmatively answered in [4], where it was shown that $\gamma = \alpha \circ \beta_\Phi : V \to \mathbb{R}^D$ satisfies the bi-Lipschitz condition if it separates orbits for any finite group G acting isometrically on V, provided that $\alpha : \mathbb{R}^{M \times N} \to \mathbb{R}^D$ selects any subset of the entries of $\beta_\Phi(v)$. Additionally, in [5], it was shown that such γ separate orbits provided that Φ is chosen generically with respect to the Zariski topology on V^N and that the right subset of entries of $\beta_\Phi(v)$ is chosen.

1.2 Results

In this paper, we generalise the results of [3, 4] to encompass arbitrary linear maps $\alpha : \mathbb{R}^{M \times N} \to \mathbb{R}^D$, which proves, in particular, that the embeddings $\gamma(\mathbf{X}) = \operatorname{diag}(\mathbf{B}^\top \operatorname{sort}(\mathbf{XA}))$ satisfy inequality (1) for generic pairs $(\mathbf{A}, \mathbf{B}) \in \mathbb{R}^{n \times D} \times \mathbb{R}^{D \times m}$ when $D > 2mn$.

Theorem 1 (Main result)**.** *The embedding $\gamma = \alpha \circ \beta_\Phi : V \to \mathbb{R}^D$ separates orbits if and only if it satisfies the bi-Lipschitz condition* (1)*.*

Moreover, we remark that, as noted in [20], the hypothesis class of functions that factor into an invariant and an optimisable part, as described at the beginning of the introduction, can contain general invariant (Lipschitz) continuous functions, if g is restricted to a sufficiently large class of functions.

Specifically, we note that any group invariant Lipschitz map into a Hilbert space factors through γ.

Theorem 2. *Let $f : V \to H$ be invariant under the action of G and Lipschitz continuous with Lipschitz constant $C_f > 0$, where H is some Hilbert space. If $\gamma = \alpha \circ \beta_\Phi : V \to \mathbb{R}^D$ separates orbits, then there exists $g : \mathbb{R}^D \to H$ Lipschitz continuous with Lipschitz constant at most C_f/c such that $f = g \circ \gamma$.*

Remark 3. It is obvious that any composition of Lipschitz continuous maps is Lipschitz continuous in turn. In particular, $f = g \circ \gamma$ is Lipschitz continuous (and invariant under the action of G) provided that $g : \mathbb{R}^D \to M$ is Lipschitz continuous, where M is a metric space.

Additionally, we note that any continuous group invariant map into a locally convex space also factors through γ.

Theorem 4. *Let $f : V \to Z$ be invariant under the action of G and continuous, where Z is some locally convex space. If $\gamma = \alpha \circ \beta_\Phi : V \to \mathbb{R}^D$ separates orbits, then there exists $g : \mathbb{R}^D \to Z$ continuous such that $g(\mathbb{R}^D)$ is a subset of the convex hull of $f(V)$ and such that $f = g \circ \gamma$.*

Remark 5. Vice versa, since compositions of continuous maps are continuous, we have that $f = g \circ \gamma$ is continuous (and invariant under the action of G) if $g : \mathbb{R}^D \to Z$ is continuous, where Z is a topological space.

1.3 Examples

Finally, we want to remark that our main result does *not* hold for general ReLU neural networks and we want to provide two examples of setups in which our results do apply.

Remark 6 (There exists an injective ReLU neural networks that is *not* bi-Lipschitz). A general ReLU neural network of depth $L \in \mathbb{N}$ is a map $\mathcal{N} : \mathbb{R}^{\ell_0} \to \mathbb{R}^{\ell_L}$ of the form

$$\mathcal{N}(x) := \begin{cases} \mathbf{W}_1 x + \mathbf{b}_1 & \text{if } L = 1, \\ \mathbf{W}_L \rho(\dots \rho(\mathbf{W}_1 x + \mathbf{b}_1) \dots) + \mathbf{b}_L & \text{if } L \geq 2, \end{cases}$$

where $\ell_0, \dots, \ell_L \in \mathbb{N}$ denote the layer widths, $\rho : \mathbb{R} \to \mathbb{R}$, $\rho(x) := \max\{x, 0\}$, denotes the ReLU activation function, which is applied component-wise to vector inputs, $\mathbf{W}_k \in \mathbb{R}^{\ell_k \times \ell_{k-1}}$ denote the weight matrices and $\mathbf{b}_k \in \mathbb{R}^{\ell_k}$ denote the bias vectors for $k \in [L]$.

A simple example of a map that is injective but not bi-Lipschitz is given by $f : \mathbb{R} \to \mathbb{R}^2$,

$$f(x) := \begin{cases} -(1, x+1) & \text{if } x < -1, \\ (x, 0) & \text{if } -1 \leq x \leq 1, \\ (1, x-1) & \text{if } 1 < x. \end{cases}$$

It is not hard to check that f is injective. At the same time, f does not satisfy the lower Lipschitz condition because $\|f(x) - f(-x)\|_2 = 2$ for $x \notin [-1, 1]$ while $|x - (-x)| = 2|x|$ is unbounded.

The above map can readily be extended to a map $f_d : \mathbb{R}^d \to \mathbb{R}^{d+1}$ for $d \in \mathbb{N}$ via $f_d(\mathbf{x}) = f_d(x_1, x_2, \dots, x_d) = (f(x_1), x_2, \dots, x_d)$. Additionally, $f = f_1$ can be implemented as a ReLU neural network with two layers: indeed, choose

$$\mathbf{W}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \qquad \mathbf{b}_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix},$$
$$\mathbf{W}_2 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \qquad \mathbf{b}_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix},$$

and note that $\mathcal{N} = f$.

Example 7 (Permutation invariant representations). Let us consider the action of the group S_m by row permutation on the vector space of matrices $\mathbb{R}^{m \times n}$. This setup is naturally encountered in learning on graphs with m vertices, where feature vectors of dimension n are associated to every vertex, if the hypothesis is to be invariant under the permutation of vertices.

We want to come back to the permutation invariant embedding proposed in [16] as a variation on the embedding proposed earlier in [3]: $\gamma_{\mathbf{A},\mathbf{B}} : \mathbb{R}^{m \times n} \to \mathbb{R}^D$,

$$\gamma_{\mathbf{A},\mathbf{B}}(\mathbf{X}) := \operatorname{diag}(\mathbf{B}^\top \operatorname{sort}(\mathbf{X}\mathbf{A})), \qquad \mathbf{X} \in \mathbb{R}^{m \times n},$$

where $\mathbf{A} \in \mathbb{R}^{n \times D}$, $\mathbf{B} \in \mathbb{R}^{m \times D}$, $\operatorname{diag} : \mathbb{R}^{D \times D} \to \mathbb{R}^D$ is the linear operator that extracts the diagonal from a $D \times D$ matrix and $\operatorname{sort} : \mathbb{R}^{m \times D} \to \mathbb{R}^{m \times D}$ denotes column-wise sorting of matrices.

In [16], it is shown that $\gamma_{\mathbf{A},\mathbf{B}}$ separates orbits for almost every $(\mathbf{A}, \mathbf{B}) \in \mathbb{R}^{n \times D} \times \mathbb{R}^{m \times D}$ provided that $D > 2mn$. Additionally, one may note that $\gamma_{\mathbf{A},\mathbf{B}}$ can be evaluated in polynomial time. Indeed, evaluating $\gamma_{\mathbf{A},\mathbf{B}}$ requires

1. a matrix multiplication of a matrix of dimension $m \times n$ and a matrix of dimension $n \times D$, which can be performed in $\mathcal{O}(Dmn)$ operations,
2. sorting D vectors of length m, which can be performed in $\mathcal{O}(Dm \log m)$ operations,
3. computing D inner products of vectors of length m, which can be performed in $\mathcal{O}(Dm)$ operations.

Overall, we have an evaluation complexity of $\mathcal{O}(Dm(n+\log m))$ which, assuming that $D \cong mn$ and $m \cong n$, is $\mathcal{O}(m^4)$.

Our main result, Theorem 1, shows that $\gamma_{\mathbf{A},\mathbf{B}}$ satisfies the bi-Lipschitz condition: there exist constants $0 < c \leq C$ such that

$$c \min_{\mathbf{P}\in S_m} \|\mathbf{X} - \mathbf{P}\mathbf{Y}\|_{\mathrm{F}} \leq \|\gamma_{\mathbf{A},\mathbf{B}}(\mathbf{X}) - \gamma_{\mathbf{A},\mathbf{B}}(\mathbf{Y})\|_2 \leq C \min_{\mathbf{P}\in S_m} \|\mathbf{X} - \mathbf{P}\mathbf{Y}\|_{\mathrm{F}},$$

for all $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{m\times n}$. Computation of an upper bound for the Lipschitz constant C is possible by Proposition 23. For practical purposes, it would also be interesting to lower bound the lower Lipschitz constant $c > 0$. While this is hard, in general, in some special cases it is possible.

Example 8 (Sign retrieval). In sign retrieval, estimating c explicitly is simplified by what is known as the σ-strong complement property [7, 8]: we are interested in the recovery of vectors $\mathbf{x} \in \mathbb{R}^n$ from magnitude-only measurements

$$|\langle \mathbf{x}, \mathbf{a}_i \rangle|, \qquad i \in [D], \tag{2}$$

where $(\mathbf{a}_i)_{i=1}^D \in \mathbb{R}^n$ is a sequence of measurement vectors. Since $\mathbf{x}$ and $-\mathbf{x}$ generate the same measurements, one typically aims to recover vectors *up to a global sign*; i.e., up to the equivalence relation $\mathbf{x} \sim \mathbf{y} :\iff \mathbf{x} = \mathbf{y}$ or $\mathbf{x} = -\mathbf{y}$; or, up to action of the group $G = \{-1, 1\}$ on the vector space of signals $\mathbb{R}^d$.

The action of the group S_2 by row permutation on the vector space of matrices $\mathbb{R}^{2\times n}$ is closely related to this sign retrieval problem as demonstrated in [6]: indeed, $\beta_{\mathbf{A}} : \mathbb{R}^{2\times n} \to \mathbb{R}^{2\times D}$,

$$\beta_{\mathbf{A}}(\mathbf{X}) := \mathrm{sort}(\mathbf{X}\mathbf{A}), \qquad \mathbf{X} \in \mathbb{R}^{2\times n},$$

where $\mathbf{A} \in \mathbb{R}^{n\times D}$ is a matrix with column vectors $(\mathbf{a}_i)_{i=1}^D \in \mathbb{R}^n$, separates orbits if and only if all $\mathbf{x} \in \mathbb{R}^n$ can be uniquely recovered from the measurements (2) up to a global sign.

However, not only the orbit separating properties of $\beta_{\mathbf{A}}$ are related to a sign retrieval problem, also the bi-Lipschitz condition is related to the Lipschitz properties of the sign retrieval operator $\mathcal{A} : \mathbb{R}^d/\{-1, 1\} \to \mathbb{R}_+^D$,

$$\mathcal{A}(\mathbf{x})_i := |\langle \mathbf{x}, \mathbf{a}_i \rangle|, \qquad i \in [D].$$

Indeed, the Lipschitz constant for the embedding $\beta_{\mathbf{A}}$ is exactly the Lipschitz constant of $\mathcal{A}$ and thus given by the largest singular value $\sigma_1(\mathbf{A})$ of the matrix $\mathbf{A}$. Moreover, the lower Lipschitz constant for the embedding $\beta_{\mathbf{A}}$ is also exactly the lower Lipschitz constant of $\mathcal{A}$ and thus exactly given by the quantity

$$c = \min_{S\subseteq[D]} \sqrt{\sigma_n^2(\mathbf{A}_S) + \sigma_n^2(\mathbf{A}_{S^c})} \tag{3}$$

as demonstrated in [6, 7]. Here, $\sigma_n(\cdot)$ is used to denote the n-th singular value (in decreasing order) of a matrix and $\mathbf{A}_S \in \mathbb{R}^{n\times|S|}$ denotes the matrix obtained by only keeping the columns whose indices are elements of $S \subseteq [D]$.

Equation (3) can allow for relatively simple computation of the lower Lipschitz constant of the permutation-invariant embedding $\beta_{\mathbf{A}}$. Indeed, consider $\mathbf{A} \in \mathbb{R}^{2\times 3}$ whose columns are given by the Mercedes-Benz frame

$$\mathbf{a}_1 = \begin{pmatrix} -\sqrt{3}/2 \\ -1/2 \end{pmatrix}, \qquad \mathbf{a}_2 = \begin{pmatrix} \sqrt{3}/2 \\ -1/2 \end{pmatrix}, \qquad \mathbf{a}_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

as an example. Then, it is a simple exercise to show that $c = \frac{1}{\sqrt{2}}$.

2 Continuous maps factor through sorting based embeddings

In this section, we prove Theorems 2 and 4. These two results are based on two well-known extension theorems: the Kirszbraun extension theorem and the Tietze extension theorem.

The precise version of the Kirszbraun extension theorem that we are going to use is due to F. A. Valentine [21].

Theorem 9 (Kirszbraun–Valentine extension theorem)**.** *Let H_1, H_2 be Hilbert spaces, let $S \subseteq H_1$ and let $g_0 : S \to H_2$ be a Lipschitz continuous map with Lipschitz constant $C > 0$. Then, g_0 can be extended to a Lipschitz continuous map $g : H_1 \to H_2$ with the same Lipschitz constant C.*

We note that the above theorem implies that any Lipschitz continuous map factors through any lower Lipschitz continuous map in the following sense.

Corollary 10. *Let X be a metric space, let H_1, H_2 be Hilbert spaces, let $h : X \to H_1$ be lower Lipschitz continuous with lower Lipschitz constant $c_h > 0$ and let $f : X \to H_2$ be Lipschitz continuous with Lipschitz constant $C_f > 0$. Then, there exists $g : H_1 \to H_2$ Lipschitz continuous with Lipschitz constant at most C_f/c_h such that $f = g \circ h$.*

Proof. Let d_X denote the metric on X and let $\|\cdot\|_{H_i}$ denote the norm on H_i where $i = 1, 2$.

Since h is lower Lipschitz continuous, it is injective and thus invertible on its range $\mathcal{R}(h)$. We may therefore define $g_0 := f \circ h^{-1} : \mathcal{R}(h) \subseteq H_1 \to H_2$. It clear that g_0 is Lipschitz continuous with Lipschitz constant at most C_f/c_h since

$$\begin{aligned} \|g_0(x) - g_0(y)\|_{H_2} &= \|f(h^{-1}(x)) - f(h^{-1}(y))\|_{H_2} \le C_f d_X(h^{-1}(x), h^{-1}(y)) \\ &\le \frac{C_f}{c_h}\|x - y\|_{H_1} \end{aligned}$$

for $x, y \in H_1$. Therefore, by the Kirszbraun–Valentine extension theorem, g_0 extends to $g : H_1 \to H_2$ Lipschitz continuous with Lipschitz constant at most C_f/c_h. Finally, we have $f = g \circ h$ by construction. □

Remark 11. Most likely this corollary has been proven before or been included in a textbook as an exercise. We were unable to find a reference, however.

We can now prove Theorem 2.

Proof of Theorem 2. Let $\|\cdot\|_H$ denote the norm on H.

Since f is invariant under the action of G, it descends through the quotient to $\widehat{f} : V/{\sim} \to H$. Moreover, since f is Lipschitz continuous with Lipschitz constant C_f, so is $\widehat{f}$: let $v, w \in V$ and let $g \in G$ be such that $\operatorname{dist}(v, w) = \|v - gw\|_V$. Then, it holds that

$$\|\widehat{f}([v]) - \widehat{f}([w])\|_H = \|f(v) - f(gw)\|_H \le C_f \|v - gw\|_V = C_f \operatorname{dist}(v, w),$$

where $[v] = \{gv \mid g \in G\}$ denotes the equivalence class of v and $[w]$ denotes the equivalence class of w.

If $\gamma : V \to \mathbb{R}^D$ separates orbits, then it satisfies the bi-Lipschitz condition according to our main theorem. In particular, since γ is also invariant under the action of G, it also descends through the quotient to $\widehat{\gamma} : V/{\sim} \to H$ and $\widehat{\gamma}$ is lower Lipschitz continuous with lower Lipschitz constant $c > 0$.

By Corollary 10, there exists $g : \mathbb{R}^D \to H$ Lipschitz continuous with Lipschitz constant at most C_f/c such that $\widehat{f} = g \circ \widehat{\gamma}$. Letting $v \in W$, we also get

$$f(v) = \widehat{f}([v]) = g(\widehat{\gamma}([v])) = g(\gamma(v)).$$

□

Similarly, the precise version of the Tietze extension theorem that we are going to use is due to J. Dugundji [15].

Theorem 12 (Dugundji–Tietze extension theorem)**.** *Let Y be a metric space, let Z be a locally convex space, let $S \subseteq Y$ be a closed subset of Y and let $g_0 : S \to Z$ be a continuous map. Then, g_0 can be extended to a continuous map $g : Y \to Z$ in such a way that $g(Y)$ is a subset of the convex hull of $g_0(S)$.*

We note that this immediately implies that any continuous function factors through any injective, relatively open function with closed range.

Corollary 13. *Let X be a topological space, let Y be a metric space and let Z be a locally convex space. Let $h : X \to Y$ be an injective, relatively open map with closed range and let $f : X \to Z$ be continuous. Then, there exists $g : Y \to Z$ continuous such that $g(Y)$ is a subset of the convex hull of $f(X)$ and such that $f = g \circ h$.*

Proof. Since h is injective, it is invertible on its range $\mathcal{R}(h)$. We may therefore define $g_0 := f \circ h^{-1} : \mathcal{R}(h) \subseteq Y \to Z$. Since f is continuous and h is relatively open, it follows that g_0 is continuous. Since the range of h is closed, the Dugundji–Tietze extension theorem implies that g_0 extends to a continuous function $g : Y \to Z$ such that $g(Y)$ is contained in the convex hull of $g_0(\mathcal{R}(h)) = f(X)$. Finally, we have $f = g \circ h$ by construction. □

Remark 14. Again, this corollary has most likely been proven or, at least, mentioned before but we were unable to find a reference.

We can now prove Theorem 4.

Proof of Theorem 4. Again, f descends through the quotient to $\widehat{f} : V/\sim \to Z$ and, since f is continuous, so is $\widehat{f}$. Because γ separates orbits, our main theorem implies that γ satisfies the bi-Lipschitz condition. Of course, γ descends through the quotient to $\widehat{\gamma} : V/\sim \to H$ and, since γ satisfies the bi-Lipschitz condition, $\widehat{\gamma}$ is bi-Lipschitz such that its range $\mathcal{R}(\widehat{\gamma})$ is closed and $\widehat{\gamma}$ is injective. Therefore, $\widehat{\gamma}$ is invertible on its range $\mathcal{R}(\widehat{\gamma})$ and its inverse is Lipschitz continuous on $\mathcal{R}(\widehat{\gamma})$: it follows that $\widehat{\gamma}$ is relatively open. By Corollary 13, there exists $g : \mathbb{R}^D \to Z$ continuous such that $g(\mathbb{R}^D)$ is a subset of the convex hull of $\widehat{f}(V/\sim) = f(V)$ and $\widehat{f} = g \circ \widehat{\gamma}$. As before, we get $f = g \circ \gamma$. □

3 Preliminaries for the proof of Theorem 1

3.1 Sorting coorbits

Denote the set of permutations that sort a vector $\mathbf{x} \in \mathbb{R}^M$ by

$$\mathcal{L}(\mathbf{x}) := \{\sigma \in S_M \mid \sigma\mathbf{x} = \operatorname{sort}(\mathbf{x})\}.$$

Let us furthermore denote the stabiliser of $\mathbf{x}$ by

$$\mathcal{H}(\mathbf{x}) := \{\sigma \in S_M \mid \sigma\mathbf{x} = \mathbf{x}\}.$$

The sets $\mathcal{L}(\mathbf{x})$ and $\mathcal{H}(\mathbf{x})$ are related in the following way.

Proposition 15. *Let $\mathbf{x} \in \mathbb{R}^M$ and $\sigma \in \mathcal{L}(\mathbf{x})$. Then, $\mathcal{L}(\mathbf{x}) = \sigma\mathcal{H}(\mathbf{x})$.*

Next, we may denote

$$\delta(\mathbf{x}) := \min_{\substack{i,j \in [M] \\ x_i \neq x_j}} |x_i - x_j|.$$

If the vector $\mathbf{x}$ is *not* constant, then $\delta(\mathbf{x})$ is exactly a scalar multiple of the distance of $\mathbf{x}$ to the next hyperplane of the form $\{\mathbf{x} \in \mathbb{R}^M \mid x_i = x_j\}$, for $i, j \in [M]$ with $i < j$ that does *not* contain $\mathbf{x}$.

Proposition 16. *Let $\mathbf{x} \in \mathbb{R}^M$. Then,*

$$\delta(\mathbf{x}) = \min_{\sigma \in S_M \setminus \mathcal{H}(\mathbf{x})} \|(\sigma - e)\mathbf{x}\|_\infty = \min_{\sigma \in S_M \setminus \mathcal{L}(\mathbf{x})} \|\sigma\mathbf{x} - \operatorname{sort}(\mathbf{x})\|_\infty.$$

Finally, let us denote the difference of the maximum and minimum of a vector $\mathbf{x}$ by

$$\Delta(\mathbf{x}) := \max_{i \in [M]} x_i - \min_{i \in [M]} x_i = \max_{i,j \in [M]} |x_i - x_j|.$$

Introducing the matrix $\mathbf{D} \in \mathbb{R}^{M(M-1)/2 \times M}$,

$$
\mathbf{D} = \begin{pmatrix}
1 & -1 & & & \\
\vdots & & -1 & & \\
\vdots & & & \ddots & \\
1 & & & & -1 \\
0 & 1 & -1 & & \\
\vdots & \vdots & & \ddots & \\
0 & 1 & & & -1 \\
 & & \vdots & & \\
0 & \cdots & 0 & 1 & -1
\end{pmatrix},
\quad
\begin{array}{l}
\left.\vphantom{\begin{matrix} 1 \\ 1 \\ 1 \\ 1 \end{matrix}}\right\} M-1 \\
\left.\vphantom{\begin{matrix} 1 \\ 1 \\ 1 \end{matrix}}\right\} M-2 \\
\vphantom{1} \\
\} \; 1
\end{array}
$$

we may write $\Delta(\mathbf{x}) = \|\mathbf{D}\mathbf{x}\|_\infty$, which shows that $\Delta(\cdot)$ satisfies the triangle inequality and is absolutely homogeneous. Finally, we note the following simple inequality.

Proposition 17. *Let $\mathbf{x} \in \mathbb{R}^M$. Then, $\delta(\mathbf{x}) \leq \Delta(\mathbf{x})$.*

Inspired by [4], we show a score of set inclusions and equalities. Let us start by proving the following simple inclusions (cf. [4, Lemma 2.5 on p. 7]).

Lemma 18. *Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^M$ be such that $\Delta(\mathbf{y}) < \delta(\mathbf{x})$. Then, $\mathcal{L}(\mathbf{x}+\mathbf{y}) \subseteq \mathcal{L}(\mathbf{x})$ and $\mathcal{H}(\mathbf{x}+\mathbf{y}) \subseteq \mathcal{H}(\mathbf{x})$.*

Proof. We show that $\mathcal{L}(\mathbf{x})^{\mathrm{c}} \subseteq \mathcal{L}(\mathbf{x}+\mathbf{y})^{\mathrm{c}}$: if $\sigma \notin \mathcal{L}(\mathbf{x})$, then we may find $i, j \in [M]$ such that $i < j$ and $x_{\sigma(i)} < x_{\sigma(j)}$. Therefore,

$$
\begin{aligned}
x_{\sigma(j)} + y_{\sigma(j)} - x_{\sigma(i)} - y_{\sigma(i)} &\geq \min_{\substack{i,j \in [M] \\ x_i \neq x_j}} |x_i - x_j| + \min_{i \in [M]} y_i - \max_{i \in [M]} y_i \\
&= \delta(\mathbf{x}) - \Delta(\mathbf{y}) > 0
\end{aligned}
$$

such that $\sigma \notin \mathcal{L}(\mathbf{x}+\mathbf{y})$. Finally, pick $\sigma \in \mathcal{L}(\mathbf{x}+\mathbf{y}) \subseteq \mathcal{L}(\mathbf{x})$ and note that $\mathcal{H}(\mathbf{x}+\mathbf{y}) = \sigma^{-1}\mathcal{L}(\mathbf{x}+\mathbf{y}) \subseteq \sigma^{-1}\mathcal{L}(\mathbf{x}) = \mathcal{H}(\mathbf{x})$ according to Proposition 15. □

Using this lemma, we can show that the set of sorting permutations does not change along certain straight line segments (cf. [4, Lemma 2.6 on p. 8]).

Lemma 19. *Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^M$ be such that $\Delta(\mathbf{y}) < \delta(\mathbf{x})$. Then, $\mathcal{L}(\mathbf{x}+t\mathbf{y}) = \mathcal{L}(\mathbf{x}+\mathbf{y})$ and $\mathcal{H}(\mathbf{x}+t\mathbf{y}) = \mathcal{H}(\mathbf{x}+\mathbf{y})$ for $t \in (0,1]$.*

Proof. $\subseteq$. Let $\sigma \in \mathcal{L}(\mathbf{x}+t\mathbf{y})$. Then, Lemma 18 implies that $\sigma \in \mathcal{L}(\mathbf{x})$. So, let $i, j \in [M]$ be such that $i < j$. There are two cases: either $x_{\sigma(i)} = x_{\sigma(j)}$ in which case

$$
\begin{aligned}
(\mathbf{x}+\mathbf{y})_{\sigma(i)} - (\mathbf{x}+\mathbf{y})_{\sigma(j)} &= y_{\sigma(i)} - y_{\sigma(j)} = t^{-1}(t y_{\sigma(i)} - t y_{\sigma(j)}) \\
&= t^{-1}\left((\mathbf{x}+t\mathbf{y})_{\sigma(i)} - (\mathbf{x}+t\mathbf{y})_{\sigma(j)}\right) \geq 0;
\end{aligned}
$$

or $x_{\sigma(i)} > x_{\sigma(j)}$ in which case

$$
(\mathbf{x}+\mathbf{y})_{\sigma(i)} - (\mathbf{x}+\mathbf{y})_{\sigma(j)} = x_{\sigma(i)} - x_{\sigma(j)} + y_{\sigma(i)} - y_{\sigma(j)} \geq \delta(\mathbf{x}) - \Delta(\mathbf{y}) > 0.
$$

Either way, $(\mathbf{x}+\mathbf{y})_{\sigma(i)} \geq (\mathbf{x}+\mathbf{y})_{\sigma(j)}$. Since $i, j \in [M]$ were arbitrary, we conclude that $\sigma \in \mathcal{L}(\mathbf{x}+\mathbf{y})$.

$\supseteq$. The reverse inclusion follows because $\mathbf{x}+t\mathbf{y} = t(\mathbf{x}+\mathbf{y})+(1-t)\mathbf{y}$ is a convex combination of $\mathbf{x}+\mathbf{y}$ and $\mathbf{x}$. More precisely, we have that $\sigma \in \mathcal{L}(\mathbf{x}+\mathbf{y}) \subseteq \mathcal{L}(\mathbf{x})$ such that

$$\begin{aligned}(\mathbf{x}+t\mathbf{y})_{\sigma(i)} &= t(\mathbf{x}+\mathbf{y})_{\sigma(i)} + (1-t)x_{\sigma(i)} \\ &\geq t(\mathbf{x}+\mathbf{y})_{\sigma(j)} + (1-t)x_{\sigma(j)} = (\mathbf{x}+t\mathbf{y})_{\sigma(j)}\end{aligned}$$

for all $i, j \in [M]$ such that $i < j$. Therefore, $\sigma \in \mathcal{L}(\mathbf{x}+t\mathbf{y})$.

The equality for the stabilisers follows from the equality for the permutations that sort by Proposition 15. $\square$

Next, we prove that the set of permutations that sort is stable on sufficiently small hypercubes (cf. [4, Lemma 2.8, items 1 and 2 on p. 10]).

Lemma 20. *Let $p \in \mathbb{N}$, $(\mathbf{x}_k)_{k=1}^p \in \mathbb{R}^M$, $(c_k)_{k=1}^p \in \mathbb{R}_+$ and $\epsilon \in (0,1)$ be such that*

$$\Delta(\mathbf{x}_{\ell+1}) < \delta\left(\sum_{k=1}^{\ell} \mathbf{x}_k\right), \qquad \ell \in [p-1],$$

$$\Delta\left(\sum_{k=1}^{p} (c_k - 1)\mathbf{x}_k\right) \leq \epsilon \cdot \delta\left(\sum_{k=1}^{p} \mathbf{x}_k\right).$$

Then, it holds that

1. $\mathcal{L}\left(\sum_{k=1}^{p} \mathbf{x}_k\right) = \mathcal{L}\left(\sum_{k=1}^{p} c_k\mathbf{x}_k\right)$, *2.* $\mathcal{H}\left(\sum_{k=1}^{p} \mathbf{x}_k\right) = \mathcal{H}\left(\sum_{k=1}^{p} c_k\mathbf{x}_k\right)$,

3. $(1-\epsilon)\cdot\delta\left(\sum_{k=1}^{p} \mathbf{x}_k\right) \leq \delta\left(\sum_{k=1}^{p} c_k\mathbf{x}_k\right) \leq (1+\epsilon)\cdot\delta\left(\sum_{k=1}^{p} \mathbf{x}_k\right)$.

Proof. $\supseteq$. We note that

$$\sum_{k=1}^{p} c_k\mathbf{x}_k = \sum_{k=1}^{p} \mathbf{x}_k + \sum_{k=1}^{p} (c_k - 1)\mathbf{x}_k, \qquad \Delta\left(\sum_{k=1}^{p} (c_k - 1)\mathbf{x}_k\right) < \delta\left(\sum_{k=1}^{p} \mathbf{x}_k\right).$$

Therefore,

$$\mathcal{L}\left(\sum_{k=1}^{p} c_k\mathbf{x}_k\right) \subseteq \mathcal{L}\left(\sum_{k=1}^{p} \mathbf{x}_k\right)$$

follows from Lemma 18. We conclude that the stabilisers satisfy the same inclusion.

$\subseteq$. Applying Lemma 18 inductively yields

$$\mathcal{L}\left(\sum_{k=1}^{p} \mathbf{x}_k\right) \subseteq \mathcal{L}\left(\sum_{k=1}^{p-1} \mathbf{x}_k\right) \subseteq \cdots \subseteq \mathcal{L}(\mathbf{x}_1).$$

So, if $\sigma \in \mathcal{L}(\sum_{k=1}^p \mathbf{x}_k)$ and $\tau \in \mathcal{L}(\sum_{k=1}^p c_k\mathbf{x}_k) \subseteq \mathcal{L}(\sum_{k=1}^p \mathbf{x}_k)$, then

$$\begin{aligned}\sigma\mathbf{x}_1 &= \mathrm{sort}(\mathbf{x}_1) = \tau\mathbf{x}_1, \\ \sigma(\mathbf{x}_1+\mathbf{x}_2) &= \mathrm{sort}(\mathbf{x}_1+\mathbf{x}_2) = \tau(\mathbf{x}_1+\mathbf{x}_2), \\ &\vdots \\ \sigma\left(\sum_{k=1}^p \mathbf{x}_k\right) &= \mathrm{sort}\left(\sum_{k=1}^p \mathbf{x}_k\right) = \tau\left(\sum_{k=1}^p \mathbf{x}_k\right),\end{aligned}$$

such that $\sigma\mathbf{x}_k = \tau\mathbf{x}_k$ for $k \in [p]$. Therefore,

$$\mathrm{sort}\left(\sum_{k=1}^p c_k\mathbf{x}_k\right) = \tau\left(\sum_{k=1}^p c_k\mathbf{x}_k\right) = \sum_{k=1}^p c_k\tau\mathbf{x}_k = \sum_{k=1}^p c_k\sigma\mathbf{x}_k = \sigma\left(\sum_{k=1}^p c_k\mathbf{x}_k\right)$$

and $\sigma \in \mathcal{L}(\sum_{k=1}^p c_k\mathbf{x}_k)$. It follows that the stabilisers satisfy the same inclusion.

Inequalities. Before proving the two inequalities, we establish the following claim.

Claim. Let $i, j \in [M]$. Then,

$$\left(\sum_{k=1}^p c_k\mathbf{x}_k\right)_i > \left(\sum_{k=1}^p c_k\mathbf{x}_k\right)_j \iff \left(\sum_{k=1}^p \mathbf{x}_k\right)_i > \left(\sum_{k=1}^p \mathbf{x}_k\right)_j.$$

Proof of the claim. First, note that $(ij) \in \mathcal{H}(\sum_{k=1}^p c_k\mathbf{x}_k)$ if and only if $(ij) \in \mathcal{H}(\sum_{k=1}^p \mathbf{x}_k)$ according to item 2. Now, assume by contradiction that

$$\left(\sum_{k=1}^p c_k\mathbf{x}_k\right)_i > \left(\sum_{k=1}^p c_k\mathbf{x}_k\right)_j, \qquad \left(\sum_{k=1}^p \mathbf{x}_k\right)_i < \left(\sum_{k=1}^p \mathbf{x}_k\right)_j$$

and consider the function $f : [0,1] \to \mathbb{R}$,

$$f(t) := \sum_{k=1}^p (1 + t(c_k - 1))\left((\mathbf{x}_k)_i - (\mathbf{x}_k)_j\right),$$

which satisfies $f(0) < 0$ and $f(1) > 0$. By the intermediate value theorem, there exists $t \in (0,1)$ such that $f(t) = 0$. By item 2 and

$$\Delta\left(\sum_{k=1}^p ((1 + t(c_k-1)) - 1)\mathbf{x}_k\right) = t\cdot\Delta\left(\sum_{k=1}^p (c_k-1)\mathbf{x}_k\right) < t\cdot\delta\left(\sum_{k=1}^p \mathbf{x}_k\right),$$

we may conclude that

$$(ij) \in \mathcal{H}\left(\sum_{k=1}^p (1 + t(c_k-1))\mathbf{x}_k\right) = \mathcal{H}\left(\sum_{k=1}^p \mathbf{x}_k\right)$$

which contradicts the assumption $(\sum_{k=1}^p \mathbf{x}_k)_i < (\sum_{k=1}^p \mathbf{x}_k)_j$. The reverse implication follows by noting that

$$(ij) \in \mathcal{H}\left(\sum_{k=1}^p \mathbf{x}_k\right) = \mathcal{H}\left(\sum_{k=1}^p c_k\mathbf{x}_k\right)$$

contradicts the assumption $(\sum_{k=1}^{p} c_k\mathbf{x}_k)_i > (\sum_{k=1}^{p} c_k\mathbf{x}_k)_j$ and exchanging the role of i and j.

Now, let $i, j \in [M]$ be such that $(\sum_{k=1}^{p} c_k\mathbf{x}_k)_i > (\sum_{k=1}^{p} c_k\mathbf{x}_k)_j$ and

$$\delta\left(\sum_{k=1}^{p} c_k\mathbf{x}_k\right) = \left(\sum_{k=1}^{p} c_k\mathbf{x}_k\right)_i - \left(\sum_{k=1}^{p} c_k\mathbf{x}_k\right)_j > 0.$$

Then, we may use the claim to see that

$$\begin{aligned}\delta\left(\sum_{k=1}^{p} c_k\mathbf{x}_k\right) &= \left(\sum_{k=1}^{p} \mathbf{x}_k\right)_i - \left(\sum_{k=1}^{p} \mathbf{x}_k\right)_j \\ &\qquad + \left(\sum_{k=1}^{p} (c_k-1)\mathbf{x}_k\right)_i - \left(\sum_{k=1}^{p} (c_k-1)\mathbf{x}_k\right)_j \\ &\geq \delta\left(\sum_{k=1}^{p} \mathbf{x}_k\right) - \Delta\left(\sum_{k=1}^{p} (c_k-1)\mathbf{x}_k\right) \geq (1-\epsilon)\cdot\delta\left(\sum_{k=1}^{p} \mathbf{x}_k\right).\end{aligned}$$

Vice versa, let $i, j \in [M]$ be such that $(\sum_{k=1}^{p} \mathbf{x}_k)_i > (\sum_{k=1}^{p} \mathbf{x}_k)_j$ and

$$\delta\left(\sum_{k=1}^{p} \mathbf{x}_k\right) = \left(\sum_{k=1}^{p} \mathbf{x}_k\right)_i - \left(\sum_{k=1}^{p} \mathbf{x}_k\right)_j > 0.$$

Using the claim once more yields

$$\begin{aligned}\delta\left(\sum_{k=1}^{p} \mathbf{x}_k\right) &= \left(\sum_{k=1}^{p} c_k\mathbf{x}_k\right)_i - \left(\sum_{k=1}^{p} c_k\mathbf{x}_k\right)_j \\ &\qquad + \left(\sum_{k=1}^{p} (1-c_k)\mathbf{x}_k\right)_i - \left(\sum_{k=1}^{p} (1-c_k)\mathbf{x}_k\right)_j \\ &\geq \delta\left(\sum_{k=1}^{p} c_k\mathbf{x}_k\right) - \Delta\left(\sum_{k=1}^{p} (c_k-1)\mathbf{x}_k\right) \\ &\geq \delta\left(\sum_{k=1}^{p} c_k\mathbf{x}_k\right) - \epsilon\cdot\delta\left(\sum_{k=1}^{p} \mathbf{x}_k\right)\end{aligned}$$

which proves the upper bound after rearrangement of the inequality. □

Finally, we show that the set of permutations that sort remains stable on sufficiently small hypercubes when we add a sufficiently small additional vector (cf. [4, Lemma 2.8, item 3 on p. 10]).

Lemma 21. *Let $p \in \mathbb{N}$, $(\mathbf{x}_k)_{k=1}^p, \mathbf{y} \in \mathbb{R}^M$ and $(c_k)_{k=1}^p \in \mathbb{R}_+$ be such that*

$$\Delta(\mathbf{x}_{\ell+1}) < \delta\left(\sum_{k=1}^{\ell} \mathbf{x}_k\right), \qquad \ell \in [p-1],$$

$$\Delta\left(\sum_{k=1}^{p}(c_k - 1)\mathbf{x}_k\right) \leq \frac{1}{2} \cdot \delta\left(\sum_{k=1}^{p} \mathbf{x}_k\right),$$

$$\Delta(\mathbf{y}) \leq \frac{1}{2} \cdot \delta\left(\sum_{k=1}^{p} \mathbf{x}_k\right).$$

Then,

$$\mathcal{L}\left(\sum_{k=1}^{p} \mathbf{x}_k + \mathbf{y}\right) = \mathcal{L}\left(\sum_{k=1}^{p} c_k\mathbf{x}_k + \mathbf{y}\right),$$

$$\mathcal{H}\left(\sum_{k=1}^{p} \mathbf{x}_k + \mathbf{y}\right) = \mathcal{H}\left(\sum_{k=1}^{p} c_k\mathbf{x}_k + \mathbf{y}\right).$$

Proof. Before proving the two inclusions, we show the following claim.

Claim. Let $i, j \in [M]$. Then,

$$\left(\sum_{k=1}^{p} \mathbf{x}_k\right)_i = \left(\sum_{k=1}^{p} \mathbf{x}_k\right)_j \implies \forall k \in [p] : (\mathbf{x}_k)_i = (\mathbf{x}_k)_j.$$

Proof of the claim. If $(\sum_{k=1}^p \mathbf{x}_k)_i = (\sum_{k=1}^p \mathbf{x}_k)_j$, then $(ij) \in \mathcal{H}(\sum_{k=1}^p \mathbf{x}_k)$. According to Lemma 18, we have

$$(ij) \in \mathcal{H}\left(\sum_{k=1}^{p} \mathbf{x}_k\right) \subseteq \mathcal{H}\left(\sum_{k=1}^{p-1} \mathbf{x}_k\right) \subseteq \cdots \subseteq \mathcal{H}(\mathbf{x}_1).$$

Therefore,

$$\begin{aligned}(\mathbf{x}_1)_i &= (\mathbf{x}_1)_j,\\ (\mathbf{x}_1 + \mathbf{x}_2)_i &= (\mathbf{x}_1 + \mathbf{x}_2)_j,\\ &\vdots\\ \left(\sum_{k=1}^{p} \mathbf{x}_k\right)_i &= \left(\sum_{k=1}^{p} \mathbf{x}_k\right)_j,\end{aligned}$$

which implies $(\mathbf{x}_k)_i = (\mathbf{x}_k)_j$ for $k \in [p]$.

$\subseteq$. Let $\sigma \in \mathcal{L}(\sum_{k=1}^p \mathbf{x}_k + \mathbf{y}) \subseteq \mathcal{L}(\sum_{k=1}^p \mathbf{x}_k)$ (according to Lemma 18) and let $i, j \in [M]$ be such that $i < j$.

We may now consider two cases: if $\sum_{k=1}^p (\mathbf{x}_k)_{\sigma(i)} = \sum_{k=1}^p (\mathbf{x}_k)_{\sigma(j)}$, then

$$\begin{aligned}\left(\sum_{k=1}^{p} c_k\mathbf{x}_k + \mathbf{y}\right)_{\sigma(i)} - \left(\sum_{k=1}^{p} c_k\mathbf{x}_k + \mathbf{y}\right)_{\sigma(j)} &= y_{\sigma(i)} - y_{\sigma(j)}\\ &= \left(\sum_{k=1}^{p} \mathbf{x}_k + \mathbf{y}\right)_{\sigma(i)} - \left(\sum_{k=1}^{p} \mathbf{x}_k + \mathbf{y}\right)_{\sigma(j)} \geq 0\end{aligned}$$

according to the claim.

If, vice versa, $\sum_{k=1}^p (\mathbf{x}_k)_{\sigma(i)} > \sum_{k=1}^p (\mathbf{x}_k)_{\sigma(j)}$, then

$$\begin{aligned}\left(\sum_{k=1}^p c_k\mathbf{x}_k + \mathbf{y}\right)_{\sigma(i)} &- \left(\sum_{k=1}^p c_k\mathbf{x}_k + \mathbf{y}\right)_{\sigma(j)} \\ &= \left(\sum_{k=1}^p c_k\mathbf{x}_k\right)_{\sigma(i)} - \left(\sum_{k=1}^p c_k\mathbf{x}_k\right)_{\sigma(j)} + y_{\sigma(i)} - y_{\sigma(j)} \\ &\geq \delta\left(\sum_{k=1}^p c_k\mathbf{x}_k\right) - \Delta(\mathbf{y}) \geq \frac{1}{2}\cdot\delta\left(\sum_{k=1}^p \mathbf{x}_k\right) - \Delta(\mathbf{y}) \geq 0,\end{aligned}$$

where we use the claim in the proof of Lemma 20 and Lemma 20 itself. So, in both cases $(\sum_{k=1}^p c_k\mathbf{x}_k + \mathbf{y})_{\sigma(i)} \geq (\sum_{k=1}^p c_k\mathbf{x}_k + \mathbf{y})_{\sigma(j)}$. Since $i, j \in [M]$ are arbitrary indices satisfying $i < j$, we can conclude that $\sigma \in \mathcal{L}(\sum_{k=1}^p c_k\mathbf{x}_k + \mathbf{y})$.

$\supseteq$. Let $\sigma \in \mathcal{L}(\sum_{k=1}^p c_k\mathbf{x}_k + \mathbf{y}) \subseteq \mathcal{L}(\sum_{k=1}^p c_k\mathbf{x}_k) = \mathcal{L}(\sum_{k=1}^p \mathbf{x}_k)$ and $\tau \in \mathcal{L}(\sum_{k=1}^p \mathbf{x}_k + \mathbf{y}) \subseteq \mathcal{L}(\sum_{k=1}^p c_k\mathbf{x}_k + \mathbf{y}) \subseteq \mathcal{L}(\sum_{k=1}^p \mathbf{x}_k)$. Then, we can show that $\sigma\mathbf{x}_k = \tau\mathbf{x}_k$ for $k \in [p]$, as in the proof of Lemma 20. Therefore,

$$\sigma\left(\sum_{k=1}^p c_k\mathbf{x}_k + \mathbf{y}\right) = \mathrm{sort}\left(\sum_{k=1}^p c_k\mathbf{x}_k + \mathbf{y}\right) = \tau\left(\sum_{k=1}^p c_k\mathbf{x}_k + \mathbf{y}\right)$$

implies $\sigma\mathbf{y} = \tau\mathbf{y}$. Finally, we have

$$\begin{aligned}\mathrm{sort}\left(\sum_{k=1}^p \mathbf{x}_k + \mathbf{y}\right) &= \tau\left(\sum_{k=1}^p \mathbf{x}_k + \mathbf{y}\right) = \sum_{k=1}^p \tau\mathbf{x}_k + \tau\mathbf{y} = \sum_{k=1}^p \sigma\mathbf{x}_k + \sigma\mathbf{y} \\ &= \sigma\left(\sum_{k=1}^p \mathbf{x}_k + \mathbf{y}\right)\end{aligned}$$

such that $\sigma \in \mathcal{L}(\sum_{k=1}^p \mathbf{x}_k + \mathbf{y})$.

The equality for the stabilisers follows immediately. ☐

3.2 Stabilisers of the group action

The stabilisers $H(v) := \{g \in G \,|\, gv = v\}$ of vectors $v \in V$ under the group action satisfy an inclusion similar to the one presented in Lemma 18

Lemma 22. *Let $v, w \in V$ be such that*

$$\|w\|_V < \frac{1}{2}\cdot\min_{g\notin H(v)} \|(e-g)v\|_V.$$

Then, $H(v+w) \subseteq H(v)$.

Proof. Let $g \notin H(v)$ and consider

$$\begin{aligned}\|g(v+w) - (v+w)\|_V &\geq \|(e-g)v\|_V - 2\|w\|_V \\ &\geq \min_{g\notin H(v)} \|(e-g)v\|_V - 2\|w\|_V > 0\end{aligned}$$

such that $g \notin H(v+w)$. ☐

4 The proof of Theorem 1

Throughout this section, we will work with the map $\widehat{\gamma} : V/{\sim} \to \mathbb{R}^D$ naturally obtained from setting $\widehat{\gamma}([v]) = \gamma(v)$ for $v \in V$. Here, $[v] = \{gv \mid g \in G\}$ denotes the equivalence class of v. Throughout the rest of this section, we will drop the brackets and denote the equivalence class by v as well.

We will start by proving that $\widehat{\gamma}$ is Lipschitz continuous.

Proposition 23 (Lipschitz continuity). *Let $K : V \to \mathbb{R}^{MN}$ denote the collection of coorbits*

$$Kv = \begin{pmatrix} \kappa_{\phi_1} v \\ \vdots \\ \kappa_{\phi_N} v \end{pmatrix}, \qquad v \in V,$$

and let $\|K\|_{\mathrm{op}} := \max_{\|v\|_V=1} \|Kv\|_2$ denote its operator norm. Then,

$$\widehat{\gamma} : (V/{\sim}, \mathrm{dist}) \to (\mathbb{R}^D, \|\cdot\|_2)$$

is Lipschitz continuous with Lipschitz constant bounded by $\|\alpha\|_{\mathrm{F}\to 2} \cdot \|K\|_{\mathrm{op}}$.

Proof. Somewhat similar to the proofs of [3, Theorem 3.9 on p. 15] and [4, Lemma 2.3 on p. 5], we let $v, w \in V$ be arbitrary and estimate

$$\begin{aligned}
&\|\gamma(v) - \gamma(w)\|_2^2 \\
&\quad = \big\|\alpha\left(\left(\mathrm{sort}(\kappa_{\phi_1} v) - \mathrm{sort}(\kappa_{\phi_1} w) \quad \dots \quad \mathrm{sort}(\kappa_{\phi_N} v) - \mathrm{sort}(\kappa_{\phi_N} v)\right)\right)\big\|_2^2 \\
&\quad \le \|\alpha\|_{\mathrm{F}\to 2}^2 \cdot \big\|\left(\mathrm{sort}(\kappa_{\phi_1} v) - \mathrm{sort}(\kappa_{\phi_1} w) \quad \dots \quad \mathrm{sort}(\kappa_{\phi_N} v) - \mathrm{sort}(\kappa_{\phi_N} v)\right)\big\|_{\mathrm{F}}^2 \\
&\quad = \|\alpha\|_{\mathrm{F}\to 2}^2 \cdot \sum_{\ell=1}^{N} \|\mathrm{sort}(\kappa_{\phi_\ell} v) - \mathrm{sort}(\kappa_{\phi_\ell} w)\|_2^2 \le \|\alpha\|_{\mathrm{F}\to 2}^2 \cdot \sum_{\ell=1}^{N} \|\kappa_{\phi_\ell}(v - w)\|_2^2 \\
&\quad = \|\alpha\|_{\mathrm{F}\to 2}^2 \cdot \|K(v - w)\|_2^2 \le \|\alpha\|_{\mathrm{F}\to 2}^2 \cdot \|K\|_{\mathrm{op}}^2 \cdot \|v - w\|_V^2.
\end{aligned}$$

Now, let $v, w \in V$ and let $g \in G$ be such that $\mathrm{dist}(v, w) = \|v - gw\|_V$. Then,

$$\begin{aligned}
\|\widehat{\gamma}(v) - \widehat{\gamma}(w)\|_2 = \|\gamma(v) - \gamma(gw)\|_2 &\le \|\alpha\|_{\mathrm{F}\to 2} \cdot \|K\|_{\mathrm{op}} \cdot \|v - gw\|_V \\
&= \|\alpha\|_{\mathrm{F}\to 2} \cdot \|K\|_{\mathrm{op}} \cdot \mathrm{dist}(v, w)
\end{aligned}$$

as desired. □

Remark 24. If we choose an orthonormal basis for V and express K as a matrix with respect to that basis, then $\|K\|_{\mathrm{op}}$ coincides with the largest singular value of that matrix.

The inductive procedure for the rest of the proof of the main theorem (Theorem 1) is adapted from [4]. The argument for the base case is also contained in [3].

Lemma 25 (Base case). *If $\widehat{\gamma} : (V/{\sim}, \mathrm{dist}) \to (\mathbb{R}^D, \|\cdot\|_2)$ is injective and* not *lower Lipschitz continuous, then there exists $z_1 \in V$, $\|z_1\|_V = 1$, at which the local lower Lipschitz constant of $\widehat{\gamma}$ vanishes; more precisely, there exist sequences $(v_i)_{i=1}^{\infty}$, $(w_i)_{i=1}^{\infty} \in V$ such that $v_i \not\sim w_i$ for $i \in \mathbb{N}$, $\lim_{i\to\infty} v_i = \lim_{i\to\infty} w_i = z_1$ and*

$$\lim_{i\to\infty} \frac{\|\gamma(v_i) - \gamma(w_i)\|_2}{\mathrm{dist}(v_i, w_i)} = 0.$$

Proof. Since we assume that $\widehat{\gamma}$ is not lower Lipschitz continuous, there exist sequences $(v_i)_{i=1}^{\infty}, (w_i)_{i=1}^{\infty} \in V$ such that $v_i \not\sim w_i$ for $i \in \mathbb{N}$ and

$$\lim_{i\to\infty} \frac{\|\gamma(v_i) - \gamma(w_i)\|_2}{\operatorname{dist}(v_i, w_i)} = 0.$$

The fraction is invariant under the transformation $(v, w) \mapsto r(w, v)$, $r > 0$, such that we can assume without loss of generality that $\|v_i\|_V \leq \|w_i\|_V = 1$. Since the unit ball in finite-dimensional vector spaces is compact, we can extract subsequences along which both $(v_i)_{i=1}^{\infty}$ and $(w_i)_{i=1}^{\infty}$ converge. Let us pass to these subsequences and define

$$f_1 := v_\infty := \lim_{i\to\infty} v_i, \qquad w_\infty := \lim_{i\to\infty} w_i.$$

We may now use the continuity of γ (which follows because $\widehat{\gamma}$ and thus γ is Lipschitz continuous) to see that

$$\|\gamma(v_\infty) - \gamma(w_\infty)\|_2 = \lim_{i\to\infty} \|\gamma(v_i) - \gamma(w_i)\|_2 \leq 2 \lim_{i\to\infty} \frac{\|\gamma(v_i) - \gamma(w_i)\|_2}{\operatorname{dist}(v_i, w_i)} = 0.$$

Therefore, $\gamma(v_\infty) = \gamma(w_\infty)$ and the injectivity of $\widehat{\gamma}$ implies that $f_1 = v_\infty = g w_\infty$ for some $g \in G$. It follows immediately that $\|f_1\|_V = \|w_\infty\|_V = 1$. We finally define $w_i' := g w_i$ and note that $v_i \not\sim w_i'$ for $i \in \mathbb{N}$, $\lim_{i\to\infty} w_i' = f_1$ and that

$$\lim_{i\to\infty} \frac{\|\gamma(v_i) - \gamma(w_i')\|_2}{\operatorname{dist}(v_i, w_i')} = \lim_{i\to\infty} \frac{\|\gamma(v_i) - \gamma(w_i)\|_2}{\operatorname{dist}(v_i, w_i)} = 0.$$

□

Before proceeding with the induction step, we prove the following lemma which is similar to the first claim in the proof of [4, Lemma 2.13 on p. 19]. It will be used in the proof of the induction step and the final theorem.

Lemma 26. *Let $p \in \mathbb{N}$, $p \leq d$, let $(f_j)_{j=1}^{p} \in V \setminus \{0\}$ be an orthogonal sequence such that*

$$\forall \ell \in [N], k \in [p-1] : \Delta(\kappa_{\phi_\ell} f_{k+1}) < \delta\left(\sum_{j=1}^{k} \kappa_{\phi_\ell} f_j\right). \tag{4}$$

Let $F := \operatorname{sp}(f_j)_{j=1}^{p} \subseteq V$ and assume that the local lower Lipschitz constant of

$$\widehat{\gamma}|_F : (F/\sim, \operatorname{dist}) \to (\mathbb{R}^D, \|\cdot\|_2)$$

vanishes at $f := f_1 + f_2 + \cdots + f_p$. Then, $\widehat{\gamma} : (V/\sim, \operatorname{dist}) \to (\mathbb{R}^D, \|\cdot\|_2)$ is not *injective at f.*

Proof. Let $(v_i)_{i\in\mathbb{N}}, (w_i)_{i\in\mathbb{N}} \in F$ be two sequences such that $v_i \not\sim w_i$ for $i \in \mathbb{N}$, $\lim_{i\to\infty} v_i = \lim_{i\to\infty} w_i = f$ and

$$\lim_{i\to\infty} \frac{\|\gamma(v_i) - \gamma(w_i)\|_2}{\operatorname{dist}(v_i, w_i)} = 0.$$

Expand the sequences into the orthogonal basis $(f_j)_{j=1}^{p}$ of F:

$$v_i = \sum_{j=1}^{p} c_{ij} f_j, \qquad w_i = \sum_{j=1}^{p} d_{ij} f_j,$$

where $c_{ij}, d_{ij} \in \mathbb{R}$ for $i \in \mathbb{N}$, $j \in [p]$.

Claim. For $i \in \mathbb{N}$ large enough, we have

$$\mathcal{L}(\kappa_{\phi_\ell} v_i) = \mathcal{L}(\kappa_{\phi_\ell} f) = \mathcal{L}(\kappa_{\phi_\ell} w_i), \qquad \ell \in [N].$$

Proof of the claim. We only show the first equality. The second equality follows in the same way: fix $\ell \in [N]$ arbitrary and note that with the notation

$$\|\kappa_{\phi_\ell}\|_{\mathrm{op}} = \max_{\substack{v \in V \\ \|v\|_V = 1}} \|\kappa_{\phi_\ell} v\|_2,$$

we may estimate

$$\begin{aligned}
\Delta\left(\sum_{j=1}^{p}(c_{ij}-1)\kappa_{\phi_\ell} f_j\right) &= \left\|\mathbf{D}\sum_{j=1}^{p}(c_{ij}-1)\kappa_{\phi_\ell} f_j\right\|_\infty \le \sqrt{2}\cdot\left\|\sum_{j=1}^{p}(c_{ij}-1)\kappa_{\phi_\ell} f_j\right\|_2 \\
&\le \sqrt{2}\cdot\sum_{j=1}^{p}|c_{ij}-1|\|\kappa_{\phi_\ell} f_j\|_2 \le \sqrt{2}\|\kappa_{\phi_\ell}\|_{\mathrm{op}}\cdot\sum_{j=1}^{p}|c_{ij}-1|\|f_j\|_V \\
&\le \sqrt{2d}\|\kappa_{\phi_\ell}\|_{\mathrm{op}}\cdot\left(\sum_{j=1}^{p}|c_{ij}-1|^2\|f_j\|_V^2\right)^{1/2} \\
&= \sqrt{2d}\|\kappa_{\phi_\ell}\|_{\mathrm{op}}\cdot\left\|\sum_{j=1}^{p}(c_{ij}-1)f_j\right\|_V = \sqrt{2d}\|\kappa_{\phi_\ell}\|_{\mathrm{op}}\cdot\|v_i - f\|_V.
\end{aligned}$$

The right-hand side tends to zero as $i \to \infty$. Therefore, there exists $I_\ell \in \mathbb{N}$ such that for all $i \ge I_\ell$,

$$\Delta\left(\sum_{j=1}^{p}(c_{ij}-1)\kappa_{\phi_\ell} f_j\right) < \delta(\kappa_{\phi_\ell} f).$$

According to equation (4) (and Lemma 20), we can conclude that $\mathcal{L}(\kappa_{\phi_\ell} v_i) = \mathcal{L}(\kappa_{\phi_\ell} f)$. Setting $I := \max_{\ell \in [N]} I_\ell$ allows us to conclude that, for all $i \ge I$ and all $\ell \in [N]$, $\mathcal{L}(\kappa_{\phi_\ell} v_i) = \mathcal{L}(\kappa_{\phi_\ell} f)$ as desired.

Now, let us pick $(\sigma_\ell)_{\ell=1}^N \in S_M$ be such that $\sigma_\ell \in \mathcal{L}(\kappa_{\phi_\ell} f)$ for $\ell \in [N]$ and note that the claim implies that

$$\begin{aligned}
&\|\gamma(v_i) - \gamma(w_i)\|_2 \\
&\quad = \left\|\alpha\left(\left(\mathrm{sort}(\kappa_{\phi_1} v_i) - \mathrm{sort}(\kappa_{\phi_1} w_i) \quad \ldots \quad \mathrm{sort}(\kappa_{\phi_N} v_i) - \mathrm{sort}(\kappa_{\phi_N} w_i)\right)\right)\right\|_2 \\
&\quad = \left\|\alpha\left(\sigma_1\kappa_{\phi_1}(v_i - w_i) \quad \ldots \quad \sigma_N\kappa_{\phi_N}(v_i - w_i)\right)\right\|_2,
\end{aligned}$$

for $i \in \mathbb{N}$ large enough. So, let us define $u_i := (v_i - w_i)/\|v_i - w_i\|_V$. Since the closed unit ball in F is compact, we can find a subsequence along which u_i converges. Let us pass to this subsequence and denote its limit by u. Then, we have $\|u\|_V = 1$ as well as

$$\begin{aligned}
\left\|\alpha\left(\sigma_1\kappa_{\phi_1} u \quad \ldots \quad \sigma_N\kappa_{\phi_N} u\right)\right\|_2 &= \lim_{i\to\infty}\left\|\alpha\left(\sigma_1\kappa_{\phi_1} u_i \quad \ldots \quad \sigma_N\kappa_{\phi_N} u_i\right)\right\|_2 \\
&= \lim_{i\to\infty}\frac{\left\|\alpha\left(\sigma_1\kappa_{\phi_1}(v_i - w_i) \quad \ldots \quad \sigma_N\kappa_{\phi_N}(v_i - w_i)\right)\right\|_2}{\|v_i - w_i\|_V} \\
&\le \lim_{i\to\infty}\frac{\|\gamma(v_i) - \gamma(w_i)\|_2}{\mathrm{dist}(v_i, w_i)} = 0
\end{aligned}$$

and thus $\alpha\left(\sigma_1\kappa_{\phi_1}u \quad \dots \quad \sigma_N\kappa_{\phi_N}u\right) = \mathbf{0}_D$.

We finally note that there exist arbitrarily small $\epsilon > 0$ such that $f \not\sim f+\epsilon u$: indeed, assume the opposite and we can find a sequence $(\epsilon_n)_{n\in\mathbb{N}} \in \mathbb{R}_+$ converging to zero and a group element $g \in G$ such that $f = gf+\epsilon_n gu$ for all $n \in \mathbb{N}$ because G is finite. Therefore, g stabilises f which implies $\epsilon_n gu = 0$ for all $n \in \mathbb{N}$: a contradiction to $\|u\|_V = 1$. So, let us pick such an ϵ which also satisfies

$$\epsilon < \min_{\ell\in[N]} \frac{\delta(\kappa_{\phi_\ell} f)}{\Delta(\kappa_{\phi_\ell} u)}.$$

Then, $\mathcal{L}(\kappa_{\phi_\ell}(f+\epsilon u)) \subseteq \mathcal{L}(\kappa_{\phi_\ell} f)$ for all $\ell \in [N]$ according to Lemma 18 and thus picking $(\sigma_\ell)_{\ell=1}^N \in S_M$ such that $\sigma_\ell \in \mathcal{L}(\kappa_{\phi_\ell}(f+\epsilon u))$ for all $\ell \in [N]$ guarantees that

$$\begin{aligned}
\gamma(f+\epsilon u) &= \alpha\left(\mathrm{sort}(\kappa_{\phi_1}(f+\epsilon u)) \quad \dots \quad \mathrm{sort}(\kappa_{\phi_N}(f+\epsilon u))\right) \\
&= \alpha\left(\sigma_1\kappa_{\phi_1}(f+\epsilon u) \quad \dots \quad \sigma_N\kappa_{\phi_N}(f+\epsilon u)\right) \\
&= \alpha\left(\sigma_1\kappa_{\phi_1} f \quad \dots \quad \sigma_N\kappa_{\phi_N} f\right) + \epsilon\cdot\alpha\left(\sigma_1\kappa_{\phi_1} u \quad \dots \quad \sigma_N\kappa_{\phi_N} u\right) \\
&= \alpha\left(\sigma_1\kappa_{\phi_1} f \quad \dots \quad \sigma_N\kappa_{\phi_N} f\right) = \alpha\left(\mathrm{sort}(\kappa_{\phi_1} f) \quad \dots \quad \mathrm{sort}(\kappa_{\phi_N} f)\right) \\
&= \gamma(f);
\end{aligned}$$

i.e. $\widehat{\gamma}$ is *not* injective at f. □

We are now ready to prove the induction step (cf. [4, Lemma 2.13 on p. 19])

Lemma 27 (Induction step)**.** *Let $p \in \mathbb{N}$, $p < d$, and let $(f_j)_{j=1}^p \in V \setminus \{0\}$ be an orthogonal sequence such that the local lower Lipschitz constant of $\widehat{\gamma} : (V/{\sim}, \mathrm{dist}) \to (\mathbb{R}^D, \|\cdot\|_2)$ vanishes at $f := f_1 + f_2 + \cdots + f_p$, $\|f_1\| = 1$ and*

$$\Delta(\kappa_{\phi_\ell} f_{k+1}) < \delta\left(\kappa_{\phi_\ell}\sum_{j=1}^k f_j\right), \tag{5}$$

$$\|f_{k+1}\|_V < \frac{1}{2}\cdot \min_{g\notin H(\sum_{j=1}^k f_j)} \left\|(g-e)\sum_{j=1}^k f_j\right\|_V, \tag{6}$$

for all $k \in [p-1]$, $\ell \in [N]$.

If $\widehat{\gamma}$ is injective, then there exists $f_{p+1} \in (\mathrm{sp}(f_j)_{j=1}^p)^\perp \setminus \{0\}$ such that the local lower Lipschitz constant of $\widehat{\gamma}$ vanishes at $f + f_{p+1}$ and

$$\Delta(\kappa_{\phi_\ell} f_{p+1}) < \delta\left(\kappa_{\phi_\ell}\sum_{j=1}^p f_j\right),$$

$$\|f_{p+1}\|_V < \frac{1}{2}\cdot \min_{g\notin H(\sum_{j=1}^p f_j)} \left\|(g-e)\sum_{j=1}^p f_j\right\|_V,$$

for all $\ell \in [N]$.

Proof. Let $(v_i)_{i=1}^\infty, (w_i)_{i=1}^\infty \in V$ be such that $v_i \not\sim w_i$ for $i \in \mathbb{N}$, $\lim_{i\to\infty} v_i = \lim_{i\to\infty} w_i = f$ and

$$\lim_{i\to\infty} \frac{\|\gamma(v_i)-\gamma(w_i)\|_2}{\mathrm{dist}(v_i, w_i)} = 0.$$

Let $(\mathbb{v}_i)_{i=1}^{\infty}, (\mathbb{w}_i)_{i=1}^{\infty} \in (\mathrm{sp}(f_j)_{j=1}^{p})^{\perp}$ denote the orthogonal projections of $(v_i)_{i=1}^{\infty}$, $(w_i)_{i=1}^{\infty}$ onto the orthogonal complement of $\mathrm{sp}(f_j)_{j=1}^{p}$. We may note that $\mathbb{v}_i \neq 0$ or $\mathbb{w}_i \neq 0$ for $i \in \mathbb{N}$ large enough: indeed, assume by contradiction that there exists a subsequence along which $\mathbb{v}_i = \mathbb{w}_i = 0$. Then, the local lower Lipschitz constant of $\widehat{\gamma}|_F : (F/\sim, \mathrm{dist}) \to (\mathbb{R}^D, \|\cdot\|_2)$ vanishes at f, where $F = \mathrm{sp}(f_j)_{j=1}^{p} \subseteq V$. Lemma 26 now implies that $\widehat{\gamma}$ is *not* injective: a contradiction.

In the following, we assume without loss of generality that $\|\mathbb{v}_i\|_V \leq \|\mathbb{w}_i\|_V \neq 0$ for $i \in \mathbb{N}$ by switching the roles of v_i and w_i if necessary. Now, write

$$v_i = \sum_{j=1}^{p} c_{ij} f_j + \mathbb{v}_i, \qquad w_i = \sum_{j=1}^{p} d_{ij} f_j + \mathbb{w}_i,$$

where $c_{ij}, d_{ij} \in \mathbb{R}$ for $i \in \mathbb{N}$, $j \in [p]$. Next, we define

$$u_i := w_i - v_i + \mathbb{v}_i = \sum_{j=1}^{p} (d_{ij} - c_{ij}) f_j + \mathbb{w}_i, \qquad i \in \mathbb{N},$$

and note that $\|u_i\|_V \geq \|\mathbb{w}_i\|_V > 0$ by the orthogonality of $(f_j)_{j=1}^{p}$ and $\mathbb{w}_i$. Finally, set

$$t_i := \frac{\epsilon}{\sqrt{2}\|u_i\|_V} \cdot \min\left\{ \min_{\ell \in [N]} \|\kappa_{\phi_\ell}\|_{\mathrm{op}}^{-1} \delta(\kappa_{\phi_\ell} f), \frac{1}{2\sqrt{2}} \cdot \min_{g \notin H(f)} \|(g-e)f\|_V \right\}$$

for $i \in \mathbb{N}$, where $\epsilon < 1$ and $\|\kappa_{\phi_\ell}\|_{\mathrm{op}} = \max_{\|v\|_V = 1} \|\kappa_{\phi_\ell} v\|_2$ denotes the operator norm of the coorbit $\kappa_{\phi_\ell} : V \to \mathbb{R}^M$ for $\ell \in [N]$.

We note that

$$\lim_{i\to\infty} \|\mathbb{v}_i\|_V^2 \leq \left(\lim_{i\to\infty} \sum_{j=1}^{p} |c_{ij} - 1|^2 \|f_j\|_V^2 + \|\mathbb{v}_i\|_V^2 \right) = \lim_{i\to\infty} \|v_i - f\|_V^2 = 0,$$
$$\lim_{i\to\infty} \|u_i\|_V \leq \lim_{i\to\infty} \|v_i - w_i\|_V + \lim_{i\to\infty} \|\mathbb{v}_i\|_V = 0.$$

It follows that $t_i \to \infty$ as $i \to \infty$. So, let us assume that $t_i > 1$ for the rest of this proof. Finally, note that $\|t_i u_i\|_V$ is constant in $i \in \mathbb{N}$. It follows that $t_i u_i$ converges along a subsequence. Similarly, $\|t_i \mathbb{v}_i\|_V = t_i \|\mathbb{v}_i\|_V \leq t_i \|u_i\|_V = \|t_i u_i\|_V$ is upper bounded by a constant in $i \in \mathbb{N}$ such that $t_i \mathbb{v}_i$ converges along a subsequence as well. Passing to these subsequences, we may write $t_i u_i \to u \neq 0$ and $t_i \mathbb{v}_i \to f_{p+1}$. Note that $\|f_{p+1}\|_V = \lim_{i\to\infty} t_i \|\mathbb{v}_i\|_V \leq \lim_{i\to\infty} t_i \|u_i\|_V = \|u\|_V$.

Claim 1. The sequences

$$v_i' = f + t_i \mathbb{v}_i, \qquad w_i' = f + t_i u_i, \qquad i \in \mathbb{N},$$

achieve lower Lipschitz constant zero.

Proof of Claim 1. The claim is proven in two steps. In the first step, we bound the denominator $\mathrm{dist}(v_i', w_i')$: consider $(g_i)_{i=1}^{\infty} \in G$ such that $\mathrm{dist}(v_i', w_i') = \|g_i v_i' - w_i'\|_V$. Since G is finite, there exists a group element $g \in G$ which occurs infinitely often. Let us pass to a subsequence along which this is the case. We claim that $g \in H(f)$: indeed, let $h \notin H(f)$ be arbitrary. Then, we have

$$\begin{aligned}
\|hv_i' - w_i'\|_V &= \|(h-e)f + t_i(h\mathbb{v}_i - u_i)\|_V \\
&\geq \|(h-e)f\|_V - t_i\|h\mathbb{v}_i - u_i\|_V \\
&\geq \min_{h \notin H(f)} \|(h-e)f\|_V - t_i(\|\mathbb{v}_i\|_V + \|u_i\|_V) \\
&\geq \min_{h \notin H(f)} \|(h-e)f\|_V - 2t_i\|u_i\|_V \\
&> 2t_i\|u_i\|_V \geq t_i\|\mathbb{v}_i - u_i\|_V = \|v_i' - w_i'\|_V \geq \operatorname{dist}(v_i', w_i'),
\end{aligned}$$

by the definition of t_i. So, $g \in H(f)$ which implies that

$$\operatorname{dist}(v_i', w_i') = t_i\|g\mathbb{v}_i - u_i\|_V = t_i\left\|g\mathbb{v}_i - \sum_{j=1}^{p}(d_{ij} - c_{ij})f_j - \mathbb{w}_i\right\|_V.$$

Now, we would like to use that $H(f) \subseteq H(\sum_{j=1}^{p-1} f_j) \subseteq \cdots \subseteq \mathcal{H}(f_1)$ which follows from Lemma 22 by inequality (6). Therefore, $g \in H(f_j)$ for all $j \in [p]$ and we have

$$\begin{aligned}
\operatorname{dist}(v_i', w_i') &= t_i\left\|g\mathbb{v}_i - \sum_{j=1}^{p}(d_{ij} - c_{ij})f_j - \mathbb{w}_i\right\|_V \\
&= t_i\left\|g\left(\sum_{j=1}^{p} c_{ij}f_j + \mathbb{v}_i\right) - \left(\sum_{j=1}^{p} d_{ij}f_j + \mathbb{w}_i\right)\right\|_V \\
&= t_i\|gv_i - w_i\|_V \geq t_i \operatorname{dist}(v_i, w_i).
\end{aligned}$$

In the second step, we consider the numerator $\|\gamma(v_i') - \gamma(w_i')\|_2$. For all $i \in \mathbb{N}$ and $\ell \in [N]$, pick arbitrary $\sigma_{i\ell} \in \mathcal{L}(\kappa_{\phi_\ell} v_i)$ and $\tau_{i\ell} \in \mathcal{L}(\kappa_{\phi_\ell} w_i)$. Then, we have

$$\begin{aligned}
&\|\gamma(v_i) - \gamma(w_i)\|_2 \\
&\quad= \left\|\alpha\left(\operatorname{sort}(\kappa_{\phi_1} v_i) - \operatorname{sort}(\kappa_{\phi_1} w_i) \quad \ldots \quad \operatorname{sort}(\kappa_{\phi_N} v_i) - \operatorname{sort}(\kappa_{\phi_N} w_i)\right)\right\|_2 \\
&\quad= \left\|\alpha\left(\sigma_{i1}\kappa_{\phi_1} v_i - \tau_{i1}\kappa_{\phi_1} w_i \quad \ldots \quad \sigma_{iN}\kappa_{\phi_N} v_i - \tau_{iN}\kappa_{\phi_1 N} w_i\right)\right\|_2 \\
&\quad= \left\|\alpha\left(\sigma_{i1}\kappa_{\phi_1}\left(\textstyle\sum_{j=1}^{p} c_{ij}f_j + \mathbb{v}_i\right) - \tau_{i1}\kappa_{\phi_1}\left(\textstyle\sum_{j=1}^{p} d_{ij}f_j + \mathbb{w}_i\right)\right.\right. \\
&\qquad\quad \left.\left.\ldots \quad \sigma_{iN}\kappa_{\phi_N}\left(\textstyle\sum_{j=1}^{p} c_{ij}f_j + \mathbb{v}_i\right) - \tau_{iN}\kappa_{\phi_N}\left(\textstyle\sum_{j=1}^{p} d_{ij}f_j + \mathbb{w}_i\right)\right)\right\|_2.
\end{aligned} \tag{7}$$

Next, we want to use that $\sigma_{i\ell}, \tau_{i,\phi} \in \mathcal{L}(\kappa_{\phi_\ell} \sum_{j=1}^{p} c_{ij}f_j)$ for all $\ell \in [N]$ and all $i \in \mathbb{N}$ large enough.

Claim 2. There exists $I \in \mathbb{N}$ such that, for all $i \geq I$ and all $\ell \in [N]$, we have

$$\mathcal{L}(\kappa_{\phi_\ell} v_i) = \mathcal{L}(\kappa_{\phi_\ell}(f + \mathbb{v}_i)) \subseteq \mathcal{L}(\kappa_{\phi_\ell} f) = \mathcal{L}\left(\kappa_{\phi_\ell}\sum_{j=1}^{p} c_{ij}f_j\right). \tag{8}$$

Proof of Claim 2. Let us fix $\ell \in [N]$ arbitrary. The first equality follows from Lemma 21 and inequality (5) once

$$\max\left\{\Delta\left(\kappa_{\phi_\ell}\sum_{j=1}^{p}(c_{ij} - 1)f_j\right), \Delta(\kappa_{\phi_\ell}\mathbb{v}_i)\right\} \leq \frac{1}{2}\cdot\delta(\kappa_{\phi_\ell} f). \tag{9}$$

As part of the proof of Lemma 26, we had shown that

$$\lim_{i\to\infty} \Delta\left(\kappa_{\phi_\ell} \sum_{j=1}^{p} (c_{ij} - 1) f_j\right) = 0.$$

Additionally, we can show that

$$\lim_{i\to\infty} \Delta(\kappa_{\phi_\ell} \mathbb{v}_i) \leq \sqrt{2} \cdot \lim_{i\to\infty} \|\kappa_{\phi_\ell} \mathbb{v}_i\|_2 \leq \sqrt{2} \|\kappa_{\phi_\ell}\|_{\mathrm{op}} \cdot \lim_{i\to\infty} \|\mathbb{v}_i\|_V = 0.$$

It follows that there exists an integer $I_\ell \in \mathbb{N}$ such that both inequalities in equation (9) are satisfed with $i \geq I_\ell$.

The second inclusion in equation 8 follows from Lemma 18 due to inequality (9). The final equality follows from Lemma 20 due to inequality (9) and inequality (5). Setting $I := \max_{\ell\in[N]} I_\ell$ finishes the proof of Claim 2.

Claim 3. There exists $I \in \mathbb{N}$ such that, for all $i \geq I$ and all $\ell \in [N]$, we have

$$\mathcal{L}(\kappa_{\phi_\ell} w_i) = \mathcal{L}(\kappa_{\phi_\ell}(f + u_i)) \subseteq \mathcal{L}(\kappa_{\phi_\ell} f). \tag{10}$$

Proof of Claim 3. The proof is almost identical to that of Claim 2 once we realise that

$$w_i = \sum_{j=1}^{p} c_{ij} f_j + u_i, \qquad i \in \mathbb{N}.$$

We will therefore omit it.

Proof of Claim 1 (continued). Passing to the sequences starting at I, we reconsider equation (7): we have

$$\sigma_{i\ell} \kappa_{\phi_\ell} \sum_{j=1}^{p} c_{ij} f_j = \mathrm{sort}\left(\kappa_{\phi_\ell} \sum_{j=1}^{p} c_{ij} f_j\right) = \tau_{i\ell} \kappa_{\phi_\ell} \sum_{j=1}^{p} c_{ij} f_j$$

and thus

$$\begin{aligned}
&\|\gamma(v_i) - \gamma(w_i)\|_2 \\
&\quad = \Big\|\alpha\Big(\sigma_{i1}\kappa_{\phi_1}\Big(\textstyle\sum_{j=1}^{p} c_{ij} f_j + \mathbb{v}_i\Big) - \tau_{i1}\kappa_{\phi_1}\Big(\textstyle\sum_{j=1}^{p} d_{ij} f_j + \mathbb{w}_i\Big) \\
&\qquad\quad \ldots \quad \sigma_{iN}\kappa_{\phi_N}\Big(\textstyle\sum_{j=1}^{p} c_{ij} f_j + \mathbb{v}_i\Big) - \tau_{iN}\kappa_{\phi_N}\Big(\textstyle\sum_{j=1}^{p} d_{ij} f_j + \mathbb{w}_i\Big)\Big)\Big\|_2 \\
&\quad = \Big\|\alpha\Big(\sigma_{i1}\kappa_{\phi_1}\mathbb{v}_i - \tau_{i1}\kappa_{\phi_1}\Big(\textstyle\sum_{j=1}^{p}(d_{ij} - c_{ij}) f_j + \mathbb{w}_i\Big) \\
&\qquad\quad \ldots \quad \sigma_{iN}\kappa_{\phi_N}\mathbb{v}_i - \tau_{iN}\kappa_{\phi_N}\Big(\textstyle\sum_{j=1}^{p}(d_{ij} - c_{ij}) f_j + \mathbb{w}_i\Big)\Big)\Big\|_2.
\end{aligned}$$

Next, we may use that $\sigma_{i\ell}, \tau_{i\ell} \in \mathcal{L}(\kappa_{\phi_\ell} f)$ for all $i \in \mathbb{N}$ and all $\ell \in [N]$ to see that

$$\begin{aligned}
t_i\|\gamma(v_i) - \gamma(w_i)\|_2 &= t_i\big\|\alpha\big(\sigma_{i1}\kappa_{\phi_1}\mathbb{v}_i - \tau_{i1}\kappa_{\phi_1} u_i \quad \ldots \quad \sigma_{iN}\kappa_{\phi_N}\mathbb{v}_i - \tau_{iN}\kappa_{\phi_N} u_i\big)\big\|_2 \\
&= \big\|\alpha\big(\sigma_{i1}\kappa_{\phi_1}(f + t_i\mathbb{v}_i) - \tau_{i1}\kappa_{\phi_1}(f + t_i u_i) \\
&\qquad \ldots \quad \sigma_{iN}\kappa_{\phi_N}(f + t_i\mathbb{v}_i) - \tau_{iN}\kappa_{\phi_N}(f + t_i u_i)\big)\big\|_2 \\
&= \big\|\alpha\big(\sigma_{i1}\kappa_{\phi_1} v_i' - \tau_{i1}\kappa_{\phi_1} w_i' \quad \ldots \quad \sigma_{iN}\kappa_{\phi_N} v_i' - \tau_{iN}\kappa_{\phi_N} w_i'\big)\big\|_2.
\end{aligned} \tag{11}$$

Finally, we want to use $\sigma_{i\ell} \in \mathcal{L}(\kappa_{\phi_\ell} v_i')$ and $\tau_{i\ell} \in \mathcal{L}(\kappa_{\phi_\ell} w_i')$ for all $i \in \mathbb{N}$ and all $\ell \in [N]$. The first inclusion is true because $\mathcal{L}(\kappa_{\phi_\ell}(f + \mathbb{v}_i)) = \mathcal{L}(\kappa_{\phi_\ell}(f + t_i\mathbb{v}_i))$ is guaranteed by Lemma 19 since

$$\begin{aligned}\Delta(\kappa_{\phi_\ell} t_i \mathbb{v}_i) = t_i \Delta(\kappa_{\phi_\ell} \mathbb{v}_i) &\leq \sqrt{2} t_i \|\kappa_{\phi_\ell} \mathbb{v}_i\|_2 \leq \sqrt{2} \|\kappa_{\phi_\ell}\|_{\mathrm{op}} t_i \|\mathbb{v}_i\|_V \\ &\leq \sqrt{2} \|\kappa_{\phi_\ell}\|_{\mathrm{op}} t_i \|u_i\|_V < \delta(\kappa_{\phi_\ell} f)\end{aligned}$$

according to the definition of $t_i > 1$. The second inclusion follows in exactly the same way.

Returning to equation (11), we have

$$\begin{aligned}t_i \|\gamma(v_i) - \gamma(w_i)\|_2 &= \left\|\alpha\left(\sigma_{i1}\kappa_{\phi_1} v_i' - \tau_{i1}\kappa_{\phi_1} w_i' \quad \ldots \quad \sigma_{iN}\kappa_{\phi_N} v_i' - \tau_{iN}\kappa_{\phi_N} w_i'\right)\right\|_2 \\ &= \left\|\alpha\left(\mathrm{sort}(\kappa_{\phi_1} v_i') - \mathrm{sort}(\kappa_{\phi_1} w_i') \quad \ldots \quad \mathrm{sort}(\kappa_{\phi_N} v_i') - \mathrm{sort}(\kappa_{\phi_N} w_i')\right)\right\|_2 \\ &= \|\gamma(v_i') - \gamma(w_i')\|_2.\end{aligned}$$

We finally conclude that

$$\lim_{i\to\infty} \frac{\|\gamma(v_i') - \gamma(w_i')\|_2}{\mathrm{dist}(v_i', w_i')} \leq \lim_{i\to\infty} \frac{\|\gamma(v_i) - \gamma(w_i)\|_2}{\mathrm{dist}(v_i, w_i)} = 0.$$

Remember that $t_i\mathbb{v}_i$ and $t_i u_i$ are bounded sequences that converge to f_{p+1} and u, respectively, as $i \to \infty$. Therefore, v_i' and w_i' are bounded sequences that converge to $f + f_{p+1}$ and $f + u$, respectively. Therefore,

$$\begin{aligned}\|\gamma(f + f_{p+1}) - \gamma(f + u)\|_2 &= \lim_{i\to\infty} \|\gamma(v_i') - \gamma(w_i')\|_2 \\ &\lesssim \lim_{i\to\infty} \frac{\|\gamma(v_i') - \gamma(w_i')\|_2}{\mathrm{dist}(v_i', w_i')} = 0.\end{aligned}$$

Now, the injectivity of $\widehat{\gamma}$ implies that $f + f_{p+1} \sim f + u$; i.e., for some $g \in G$, we have $g(f + f_{p+1}) = f + u$. It follows that $g \in H(f)$: indeed, assume, by contradiction, that $g \notin H(f)$. Then, we have

$$\begin{aligned}0 = \|g(f + f_{p+1}) - (f + u)\|_V &\geq \|(g - e)f\|_V - \|g f_{p+1} - u\|_V \\ &\geq \min_{g \notin H(f)} \|(g - e)f\|_V - \|f_{p+1}\|_V - \|u\|_V \\ &\geq \min_{g \notin H(f)} \|(g - e)f\|_V - 2\|u\|_V > 0\end{aligned}$$

because

$$\|u\|_V = \lim_{i\to\infty} t_i \|u_i\|_V \leq \frac{1}{4} \cdot \min_{g \notin H(f)} \|(g - e)f\|_V.$$

Therefore, $g f_{p+1} = u$ and thus $\|f_{p+1}\|_V = \|u\|_V > 0$, which shows that $f_{p+1} \neq 0$. Finally, let $w_i'' := g^{-1} w_i' = f + t_i g^{-1} u_i$. Then, the sequences $(v_i')_{i=1}^{\infty}$, $(w_i'')_{i=1}^{\infty}$ still achieve lower Lipschitz constant zero. Therefore, they achieve local lower Lipschitz constant zero at $f + f_{p+1}$. Now, f_{p+1} is orthogonal to $(f_j)_{j=1}^{k}$ and satisfies

$$\begin{aligned}\|f_{p+1}\|_V = \lim_{i\to\infty} t_i \|\mathbb{v}_i\|_V &\leq \lim_{i\to\infty} t_i \|u_i\|_V \\ &= \frac{\epsilon}{\sqrt{2}} \cdot \min\left\{\min_{\ell \in [N]} \|\kappa_{\phi_\ell}\|_{\mathrm{op}}^{-1} \delta(\kappa_{\phi_\ell} f), \frac{1}{2\sqrt{2}} \cdot \min_{g \notin H(f)} \|(g - e)f\|_V\right\}\end{aligned}$$

Therefore,

$$\begin{aligned}\Delta(\kappa_{\phi_\ell} f_{p+1}) &\leq \sqrt{2} \cdot \|\kappa_{\phi_\ell} f_{p+1}\|_2 \leq \sqrt{2}\|\kappa_{\phi_\ell}\|_{\mathrm{op}} \cdot \|f_{p+1}\|_V \\ &< \|\kappa_{\phi_\ell}\|_{\mathrm{op}} \cdot \min_{k\in[N]} \|\kappa_{\phi_k}\|_{\mathrm{op}}^{-1} \delta(\kappa_{\phi_k} f) \leq \delta(\kappa_{\phi_\ell} f)\end{aligned}$$

and the lemma is proven. □

Combining the base case and the induction step allows us to prove that injectivity implies bi-Lipschitz.

Proof of Theorem 1. Remember that $\widehat{\gamma}$ is Lipschitz continuous according to Proposition 23. Now, assume by contradiction that $\widehat{\gamma}$ is *not* lower Lipschitz continuous. Then, Lemma 25 together with Lemma 27 show that there exists an orthogonal basis $(f_j)_{j=1}^d \in V$ such that the local lower Lipschitz constant of $\widehat{\gamma}$ vanishes at $f := f_1 + f_2 + \cdots + f_d$ and

$$\forall \ell \in [N], k \in [d-1] : \Delta(\kappa_{\phi_\ell} f_{k+1}) < \delta\left(\sum_{j=1}^{k} \kappa_{\phi_\ell} f_j\right).$$

It follows from Lemma 26 that $\widehat{\gamma}$ is *not* injective: a contradiction. □

References

[1] Benjamin Aslan, Daniel Platt, and David Sheard. Group invariant machine learning by fundamental domain projections. In *Proceedings of the 1st NeurIPS Workshop on Symmetry and Geometry in Neural Representations*, volume 197 of *Proceedings of Machine Learning Research*, pages 181–218, 2023. `https://proceedings.mlr.press/v197/aslan23a/aslan23a.pdf`.

[2] Aharon Azulay and Yair Weiss. Why do deep convolutional networks generalize so poorly to small image transformations? *Journal of Machine Learning Research*, 20(184):1–25, 2019. `http://jmlr.org/papers/v20/19-519.html`.

[3] Radu Balan, Naveed Haghani, and Maneesh Singh. Permutation invariant representations with applications to graph deep learning. `https://doi.org/10.48550/arXiv.2203.07546`, March 2022.

[4] Radu Balan and Efstratios Tsoukanis. G-invariant representations using coorbits: bi-Lipschitz properties. `https://doi.org/10.48550/arXiv.2308.11784`, August 2023.

[5] Radu Balan and Efstratios Tsoukanis. G-invariant representations using coorbits: Injectivity properties. `https://doi.org/10.48550/arXiv.2310.16365`, October 2023.

[6] Radu Balan and Efstratios Tsoukanis. Relationships between the phase retrieval problem and permutation invariant embeddings. In *2023 International Conference on Sampling Theory and Applications (SampTA)*, New Haven, CT, USA, 10–14 July 2023. IEEE. `https://doi.org/10.1109/SampTA59647.2023.10301202`.

[7] Radu Balan and Yang Wang. Invertibility and robustness of phaseless reconstruction. *Applied and Computational Harmonic Analysis*, 38(3):469–488, May 2015. https://doi.org/10.1016/j.acha.2014.07.003.

[8] Afonso S. Bandeira, Jameson Cahill, Dustin G. Mixon, and Aaron A. Nelson. Saving phase: Injectivity and stability for phase retrieval. *Applied and Computational Harmonic Analysis*, 37(1):106–125, July 2014. https://doi.org/10.1016/j.acha.2013.10.002.

[9] Ben Blum-Smith and Soledad Villar. Machine learning and invariant theory. *Notices of the American Mathematical Society*, 70(8):1205–1213, September 2023. https://doi.org/10.1090/noti2760.

[10] Jameson Cahill, Andres Contreras, and Andres Contreras Hip. Stable separation of orbits for finite abelian group actions. *Journal of Fourier Analysis and Applications*, 30(12), 2024. https://doi.org/10.1007/s00041-024-10069-z.

[11] Jameson Cahill, Joseph W. Iverson, and Dustin G. Mixon. Towards a bilipschitz invariant theory. *Applied and Computational Harmonic Analysis*, 72, September 2024. https://doi.org/10.1016/j.acha.2024.101669.

[12] Jameson Cahill, Joseph W. Iverson, Dustin G. Mixon, and Daniel Packer. Group-invariant max filtering. *Foundations of Computational Mathematics*, May 2024. https://doi.org/10.1007/s10208-024-09656-9.

[13] Taco Cohen and Max Welling. Group equivariant convolutional networks. In *Proceedings of The 33rd International Conference on Machine Learning*, volume 48 of *Proceedings of Machine Learning Research*, pages 2990–2999, New York, New York, USA, 20–22 June 2016. http://proceedings.mlr.press/v48/cohenc16.pdf.

[14] N. Dalal and B. Triggs. Histograms of oriented gradients for human detection. In *2005 IEEE Computer Society Conference on Computer Vision and Pattern Recognition (CVPR'05)*, San Diego, CA, USA, 20–25 June 2015. IEEE. https://doi.org/10.1109/CVPR.2005.177.

[15] J. Dugundji. An extension of Tietze's theorem. *Pacific Journal of Mathematics*, 1(3):353–367, September 1951. https://doi.org/10.2140/pjm.1951.1.353.

[16] Nadav Dym and Steven J. Gortler. Low-dimensional invariant embeddings for universal geometric learning. *Foundations of Computational Mathematics*, February 2024. https://doi.org/10.1007/s10208-024-09641-2.

[17] Y. Lecun, L. Bottou, Y. Bengio, and P. Haffner. Gradient-based learning applied to document recognition. *Proceedings of the IEEE*, 86(11):2278–2324, November 1998. https://doi.org/10.1109/5.726791.

[18] D. G. Lowe. Object recognition from local scale-invariant features. In *Proceedings of the Seventh IEEE International Conference on Computer Vision*, Kerkyra, Greece, 20–27 September 1999. IEEE. https://doi.org/10.1109/ICCV.1999.790410.

[19] Dustin G. Mixon and Yousef Qaddura. Stable coorbit embeddings of orbifold quotients. https://doi.org/10.48550/arXiv.2403.14042, March 2024.

[20] Efstratios Tsoukanis. *G-invariant representations using coorbits*. Dissertation, University of Maryland, College Park, MD, USA, 2024. https://doi.org/10.13016/dovm-l2k4.

[21] F. A. Valentine. A Lipschitz condition preserving extension for a vector function. *American Journal of Mathematics*, 67(1):83–93, January 1945. https://doi.org/10.2307/2371917.

[22] Soledad Villar, David W. Hogg, Kate Storey-Fisher, Weichi Yao, and Ben Blum-Smith. Scalars are universal: Equivariant machine learning, structured like classical physics. In *Advances in Neural Information Processing Systems 34 (NeurIPS 2021)*, 2021.

[23] Dmitry Yarotsky. Universal approximations of invariant maps by neural networks. *Constructive Approximation*, 55:407–474, 2022. https://doi.org/10.1007/s00365-021-09546-1.