G-Invariant Representations using Coorbits: Injectivity Properties

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10/24/2023

Abstract

Consider a real vector space $\mathscr{V}$ and a finite group G acting unitarily on $\mathscr{V}$. We study the general problem of constructing a stable embedding whose domain is the quotient of the vector space modulo the group action, and whose target space is a Euclidean space. We construct an embedding Ψ and we study under which assumptions Ψ is injective in the quotient vector space. The embedding scheme we introduce is based on selecting a fixed subset from the sorted orbit $\downarrow \langle U_g w_i, x\rangle_{g\in G}$, where w_i are appropriate vectors.

1 Introduction

Machine learning techniques have impressive results when we feed them with large sets of data. In some cases, our training set can be small but we know that there are some underlying symmetries in the data structure. For example, in graph theory problems each graph is being represented as an adjacent matrix of the labeled nodes of the graph; any relabeling of the nodes shouldn't change the output of our classification or regression algorithm.

A possible solution for this problem is to increase our training set by adding, for each data point of the set, the whole orbit generated by the group action. One problem that arises is that it is computationally costly to find such highly symmetric function.

Another solution is to embed our data into an Euclidean space $\mathbb{R}^m$ with a symmetry-invariant embedding Ψ and then use $\mathbb{R}^m$ as our feature space. It is not enough for our embedding to be symmetric invariant, it should also separate data orbits. Finally, we require certain stability conditions so that small perturbations don't affect our predictions. This problem is an instance of *invariant machine learning* [19, 3, 15, 10, 20, 28, 14, 16, 21].

The most common group action in invariant machine learning are permutations [25, 11, 7] reflections [22] and translations [18]. Also, there are very interesting results in the case of equivariant machine learning [24, 20, 27, 26, 9].

Our work is influenced by [15] where it is shown that $m \approx 2d$ separating invariants are enough for an orbit-separating embedding, and by [12, 23] where the *max filter* is introduced. We work with a generalization of the *max filter*: instead of choosing the maximum element of the orbit we choose other subsets of orbit. The problem of finding permutation invariant embeddings seems to be closely connected to the phase retrieval problem where there already are a lot of important results [5, 6, 2, 1, 4, 17].

In the first chapter, we introduce our embedding scheme.

In the second chapter, we investigate and construct an injective embedding for the case of a finite subset of a vector space $\mathscr{V}$.

Finally, in the third chapter, we present an injective Coorbit embedding for a d-dimensional vector space $\mathscr{V}$.

1.1 Notation

Let $(\mathscr{V}, \langle\cdot,\cdot\rangle)$ be a d-dimensional real vector space, where $d \geq 2$. Assume $(G, \cdot)$ is a finite group of order $|G| = N$ acting unitarily on $\mathscr{V}$. For every $g \in G$, we denote by $U_g x$ the group action. On $\hat{\mathscr{V}} = \mathscr{V}/\sim$, the quotient space with respect to action of group G, we denote by $[x]$ the orbit of vector x, i.e. $[x] = \{U_g x : g \in G\}$. Consider now the natural metric, $\mathrm{d} : \hat{\mathscr{V}} \times \hat{\mathscr{V}} \to \mathbb{R}$, where

$$\mathrm{d}([x],[y]) = \min_{h_1,h_2 \in G} \|U_{h_1}x - U_{h_2}y\| = \min_{g \in G} \|x - U_g y\|.$$

Our goal is to construct a bi-Lipschitz Euclidean embedding on the metric space $(\hat{\mathscr{V}}, d)$. Specifically, we want to construct a function $\Psi : \mathscr{V} \to \mathbb{R}^m$ such that

1. $\Psi(U_g x) = \Psi(x),\ \forall x \in \mathscr{V},\ \forall g \in G,$

2. If $x, y \in \mathscr{V}$ are such that $\Psi(x) = \Psi(y)$, then there exist $g \in G$ such that $y = U_g x$,

3. There are $0 < a < b < \infty$ such that for any $x, y \in \mathscr{V}$

$$a\, \mathrm{d}([x],[y])^2 \leq \|\Psi(x) - \Psi(y)\|^2 \leq b(\mathrm{d}([x],[y]))^2.$$

The invariance property (1) lifts Ψ to a map $\hat{\Psi}$ acting on the quotient space $\hat{\mathscr{V}} = \mathscr{V}/\sim$, where $x \sim y$ if and only if $y = U_g x$ for some $g \in G$:

$$\hat{\Psi} : \hat{\mathscr{V}} \to \mathbb{R}^m, \quad \hat{\Psi}([x]) = \Psi(x), \quad \forall [x] \in \hat{\mathscr{V}}.$$

If a G-invariant map Ψ satisfies property (2) we say that Ψ separates the G-orbits in $\mathbb{R}^d$.

Our construction for the embedding Ψ is based on a non-linear sorting map.

Definition 1.1. *Let $\downarrow : \mathbb{R}^r \to \mathbb{R}^r$ be the operator that takes as input a vector in $\mathbb{R}^r$ and returns a sorted, in decreasing order, vector of length r with same entries as input vector.*

For a number $p \in \mathbb{N}$, fix a p-tuple of vectors $\mathbf{w} = (w_1, \ldots, w_p) \in \mathscr{V}^p$. For any $i \in [p]$ and $j \in [N]$ we define the operator $\Phi_{w_i,j} : \mathscr{V} \to \mathbb{R}$ so that $\Phi_{w_i,j}(x)$ is the j-th coordinate of vector $\downarrow \langle U_g w_i, x\rangle_{g\in G}$. Now fix a set $S \subset [N] \times [p]$ such that $|S| = m$, and for $i \in [p]$, set $S_i = \{k \in [N] : (k,i) \in S\}$. We denote by m_i the cardinality of the set S_i, thus $m = \sum_{i=1}^{p} m_i$. Let $\ell : \mathbb{R}^m \to \mathbb{R}^{2d}$ be a linear transformation and consider the map,

$$\Psi = \Psi_{\mathbf{w},S,\ell} = \ell \circ \Phi_{\mathbf{w},S} : \mathscr{V} \to \mathbb{R}^{2d}$$

with

$$\Phi_{\mathbf{w},S}(x) = [\downarrow\{\Phi_{w_1,j}(x)\}_{j\in S_1}, \ldots, \downarrow\{\Phi_{w_p,j}(x)\}_{j\in S_p}] \in \mathbb{R}^m. \tag{1}$$

Therefore, our proposal for constructing a stable embedding is the function Ψ of the form

$$\Psi(x) = \Psi_{\mathbf{w},S,\ell}(x) = \ell(\Phi_{\mathbf{w},S}(x)).$$

For the rest of the paper when the p-tuple of vectors $\mathbf{w}$ is clearly implied we will denote by $\Phi_{i,j}$ the $\Phi_{w_i,j}$. Also by $\{g_1, \ldots, g_N\}$, we will denote an arbitrarily, but fixed, enumeration of the group G.

1.2 Semialgebraic geometry notation

In this section we will follow the notation of [13].

Definition 1.2. *An affine algebraic variety is the set of common zeros over an algebraically closed field k of some family of polynomials.*

Remark 1.3. *In literature sometimes in the definition of affine variety is required the ideal generated by defining polynomials to be prime. In this paper we will call that case irreducible variety.*

A generalization of algebraic sets is found in semialgebraic sets, which encompass polynomial inequalities in addition to algebraic equations.

Definition 1.4. *Let $\mathbb{F}$ be a real closed field. A subset S of $\mathbb{F}^n$ is a "semialgebraic set" if it is a finite union of sets defined by polynomial equalities of the form $\{(x_1, ..., x_n) \in \mathbb{F}^n \mid P(x_1, ..., x_n) = 0\}$ and of sets defined by polynomial inequalities of the form $\{(x_1, ..., x_n) \in \mathbb{F}^n \mid Q(x_1, ..., x_n) > 0\}$.*

Definition 1.5. *Let X, Y be two varieties. A continuous map $f : X \to Y$ is called morphism if $\forall p \in X$ there is a Zariski open set U containing p and polynomials functions g and g such that $\forall q \in U$, $f(q) = \frac{g(q)}{h(q)}$ and $h(q) \neq 0$.*

Now we will state some results from [13] without proof.

Proposition 1.6 (Proposition 2.15 in [13])**.** *A semialgebraic set A can be decomposed as the disjoint union of finitely many pieces which are semialgebraically homeomorphic to open hypercubes $(0, 1)^{d_i}$ of different dimensions.*

Definition 1.7. *Let A be decomposed as the disjoint union of finitely many pieces which are semialgebraically homeomorphic to open hypercubes $\{(0, 1)^{d_i}\}_{i \in I}$. Then we define the dimension of A to be the maximum dimension of hupercubes $(0, 1)^{d_i}$, i.e. $\dim(A) = \max_{i \in I} d_i$.*

Two corollaries of Tarski-Seidenberg theorem are the following:

Corollary 1.8 (Corollary 2.4 in [13])**.** *If A is a semialgebraic subset of $\mathbb{R}^{n+k}$, its image by the projection on the space of the first n coordinates is a semialgebraic subset of $\mathbb{R}^n$.*

Corollary 1.9 (Corollary 2.5 in [13])**.** *If A is a semialgebraic subset of $\mathbb{R}^n$, its closure in $\mathbb{R}^n$ is again semialgebraic.*

Let $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$ be semialgebraic sets. A mapping $f : A \to B$ is called semialgebraic if its graph:

$$\Gamma_f = \{(x, y) \in A \times B : y = f(x)\}$$

is a semialgebraic set of $\mathbb{R}^m \times \mathbb{R}^n$.

Proposition 1.10 (Corollary 2.9 and 2.2.1 in [13])**.** *1. If $f : A \to B$ is a morphism, then it is also semialgebraic.*

2. The direct image and the inverse image of a semialgebraic set by a semialgebraic mapping are semialgebraic.

3. The composition of two semialgebraic mappings is semialgebraic.

A simple corollary of Corollary 1.8 anf Proposition 1.10(1),(2) is the following:

Corollary 1.11. *Let $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$ be semialgebraic sets and $f :$ $A \to B$ be a morphism. Then $f(A)$ is also an algebraic set.*

Finally two very important theorems of semialgebraic geometry are the following:

Theorem 1.12 (Theorem 3.18 in [13])**.** *Let A be a semialgebraic subset of $\mathbb{R}^n$, and $f : A \to R^k$ a semialgebraic mapping (not necessarily continuous). Then* $\dim f(A) \leq \dim A$.

Theorem 1.13 (Theorem 3.20 in [13])**.** *Let $A \subset \mathbb{R}^n$ be a semialgebraic set. Its dimension as a semialgebraic set is equal to the dimension, as an algebraic set, of its Zariski closure $\bar{A}^S$.*

A simple corollary of Theorem 1.13 is the following:

Corollary 1.14. *Let a semialgebraic subset $A \subset \mathbb{R}^N$. If* $\dim(A) < n$ *then A is nowhere dense.*

Morevoer, note that any semiaglebraic set consists of finitely many connected components.

Theorem 1.15 (Theorem 2.23 in [13])**.** *Every semialgebraic set has finitely many connected components which are semialgebraic. Every semialgebraic set is locally connected.*

Finally, a very usefull Corolarry of *"Hardt's semialgebraic triviality"* is the following.

Corollary 1.16 (Corolarry 4.2 in [13])**.** *Let $A \subset \mathbb{R}^n$ be a semialgebraic set and $f : A \to \mathbb{R}^k$ a continuous semialgebraic mapping. For $d \in \mathbb{N}$, the set*

$$\{b \in \mathbb{R}^k : \dim(f^{-1}(b)) = d\}$$

is a semialgebraic subset of $\mathbb{R}^k$ of dimension not greater than $\dim(A) - d$.

2 Representations of finite subsets of inner product spaces

The first case we examine is when $\mathscr{A}$ is a finite subset of a real vector space $\mathscr{V}$. We also assume that $\mathscr{A}$ is G-invariant, meaning that for every $x \in \mathscr{A}$ and for every $g \in G$, $U_g x$ is also in $\mathscr{A}$.

Theorem 2.1. *Let G be a finite subgroup of $O(d)$ and $\mathscr{A}$ a finite G-invariant subset of an inner product space $\mathscr{V}$. Then, for a generic $w \in \mathscr{V}$ (with respect to the Zariski topology) and any fixed $j \in [N]$, the map $\Phi_{w,j}$ is injective on the quotient space $\hat{\mathscr{A}}$ and bi-Lipschitz.*

Proof. For fixed $x, y \in \mathscr{A}$, let

$$\mathscr{W}_{x,y} = \bigcup_{h_1,h_2 \in G} \{U_{h_1}x - U_{h_2}y\}^{\perp}$$

and

$$\mathscr{W} = \bigcup_{\substack{x,y \in \mathscr{A} \\ x \nsim y}} W_{x,y} = \bigcup_{\substack{x,y \in \mathscr{A} \\ x \nsim y}} \bigcup_{h_1,h_2 \in G} \{U_{h_1}x - U_{h_2}y\}^{\perp}.$$

Given $i \in [N]$ and $w \in \mathscr{V}$, recall that $\Phi_{w,j}(x)$ is the j-th coordinate of vector $\downarrow \langle U_g w, x\rangle_{g \in G}$. From the definition of the set $\mathscr{W}$ we notice that for any vector $w \in \mathscr{W}^c$ the operator ϕ_w^i separates different orbits of elements of $\mathscr{A}$.

Notice that $\mathscr{W}$ is a finite union of $(d-1)$-dimensional subspaces, making it a closed set with zero measure and nowhere dense with zero Lebesgue measure in $\mathscr{V}$. Consequently, for a generic element $w \in \mathscr{V}$ with respect to the Zariski topology, it provides an injective embedding $\phi_w^j(x)$. However, we still need to demonstrate that if the map $\phi_w^j(x)$ is injective, it is also bi-Lipschitz. That is to find $a_w, b_w \in \mathbb{R}$ with $0 < a_w \leq b_w$ such that for all $x, y \in \mathscr{A}$

$$a_w \, \mathrm{d}(x,y) \leq |\phi_w^j(x) - \phi_w^j(y)| \leq b_w \, \mathrm{d}(x,y).$$

As the set $\mathscr{A}$ is finite so is $\mathscr{A} \times \mathscr{A}$. Hence, $\{\mathrm{d}(x,y) | x, y \in \mathscr{A},\ x \nsim y\}$ is a finite set of positive numbers.

The optimal "bi-Lipschitz constants" are

$$a_w = \min_{\substack{x,y \in \mathscr{A} \\ x \nsim y}} \frac{|\phi_w^j(x) - \phi_w^j(y)|}{\mathrm{d}(x,y)} = \min_{\substack{x,y \in \mathscr{A} \\ x \nsim y}} \frac{\left| \max\limits_{g \in G} \langle U_g w, x \rangle - \max\limits_{g \in G} \langle U_g w, y \rangle \right|}{\min_{g \in G} \|U_g x - y\|}$$

and

$$b_w = \max_{\substack{x,y \in \mathscr{A} \\ x \nsim y}} \frac{|\phi_w^j(x) - \phi_w^j(x)|}{\mathrm{d}(x,y)} = \max_{\substack{x,y \in \mathscr{A} \\ x \nsim y}} \frac{\left| \max\limits_{g \in G} \langle U_g w, x \rangle - \max\limits_{g \in G} \langle U_g w, y \rangle \right|}{\min_{g \in G} \|U_g x - y\|}.$$

□

Notice that the upper Lipschitz bound above is sharp. However, if we don't require sharpness, there is a way to find an easily computable upper Lipschitz bound in the following manner:

Without loss of generality, suppose that

$$\Phi_{w,j}(x) \geq \Phi_{w,j}(y).$$

Let $g_j^x, g_j^y \in G$ such that $\Phi_{w,j}(x) = \langle w, U_{g_j^x} x \rangle$ and $\Phi_{w,j}(y) = \langle w, U_{g_j^y} x \rangle$, respectively, and take $g_0 \in G$ satisfying $\mathrm{d}(x,y) = \|x - U_{g_0} y\|$. Then, from the pigeonhole principle there exists $k \leq j$ such that $\langle w, U_{g_k^x g_0} y \rangle \leq \langle w, U_{g_j^y} y \rangle$. Then, we have

$$\begin{aligned}
|\Phi_{w,j}(x) - \Phi_{w,j}(y)| &= \langle w, U_{g_j^x} x \rangle - \langle w, U_{g_j^y} y \rangle \\
&\leq \langle w, U_{g_k^x} x \rangle - \langle w, U_{g_k^x} U_{g_0} y \rangle \\
&= \langle w, U_{g_k^x} (x - U_{g_0} y) \rangle \\
&\leq \|w\| \|x - U_{g_0} y\| \\
&= \|w\| \, \mathrm{d}(x,y).
\end{aligned}$$

Therefore, $b_w = \|w\|$ is a also upper Lipschitz bound.

3 Representation of inner product spaces

Fix $\mathbf{w} = (w_1, \ldots, w_p)$ and take $S = \{(1,1), \ldots, (1,p)\} \subset [N] \times [p]$. Recall that $S_i = \{k \in [N] : (k,i) \in S\}$. In that case $\forall i \in [p]$, $S_i = \{1\}$, so $\Phi_{\mathbf{w},S}$ is the *max filter* map $(\langle\langle w_1, x\rangle\rangle, \ldots, \langle\langle w_m, x\rangle\rangle)^T$, where

$$\begin{aligned}\langle\langle w_i, x\rangle\rangle &= \sup_{g_1,g_2\in G} \langle U_{g_1} w_i, U_{g_2} x\rangle = \max_{g_1,g_2\in G} \langle U_{g_1} w_i, U_{g_2} x\rangle \\ &= \max_{g\in G} \langle U_g w_i, x\rangle = \max_{g\in G} \langle w_i, U_g x\rangle.\end{aligned}$$

In [11], it is shown that $2d$ vectors are enough for the construction of an injective embedding.

Theorem 3.1 ([11, Lemma 12])**.** *Consider any finite subgroup $G \leq O(d)$. For a generic $\boldsymbol{w} \in \mathscr{V}^p$ and for $S = \{(1,1), \ldots, (1,p)\}$, the map $\Phi_{\boldsymbol{w},S}$ separates G-orbits in $\mathbb{R}^d$ provided that $p \geq 2d$.*

Our goal is to examine the pairs $(\mathbf{w}, S)$ where $\mathbf{w} \in \mathscr{V}^p$ and S is subset of $[N] \times [p]$ with $m = |S|$ such that $\widehat{\Phi^A_{\mathbf{w}}} : \hat{\mathscr{V}} \to \mathbb{R}^m$ is injective. In other words, we are interested in all the pairs $(\mathbf{w}, S)$ for which the following equivalence holds for all $x, y \in \mathscr{V}$:

$$\Phi_{\mathbf{w},S}(x) = \Phi_{\mathbf{w},S}(y) \iff [x] = [y]. \tag{2}$$

In our next Theorem 3.2, we generalize Theorem 3.1; we show that one can replace the maximum element of the orbit $\langle U_g w, \cdot\rangle$ with any other fixed element of that same orbit.

Theorem 3.2. *Let $p \geq 2d$ and $S \subset [N] \times [p]$. Suppose that $\forall i \in [p]$, $S_i \neq \emptyset$. Then, for a generic with respect to Zariski topology $\boldsymbol{w} \in \mathscr{V}^p$, the map $\Phi_{\boldsymbol{w},S}$ is injective.*

Before we are able to prove Theorem 3.2 we need some additional notation and certain lemmas. Let $\mathscr{V}$ be an inner product space of dimension d, and $G \leq O(d)$ a finite subgroup of the group of orthogonal transformations on $\mathscr{V}$. For a fixed $w \in \mathscr{V}$ and $i \in [N]$, recall that $\Phi_{w,j}(x)$ represents the j-th coordinate of $\downarrow(\langle U_g w, x\rangle g \in G)$. It's important to note that $\Phi w, j$ satisfies specific scaling and symmetry properties, which we state in the form of a lemma:

Lemma 3.3. *For j, λ and $\mathscr{V}$ as above,*

$$\Phi_{\lambda w,j}(x) = \Phi_{w,j}(\lambda x) = \lambda \Phi_{w,j}(x), \qquad \forall w, x \in \mathscr{V}, \quad \lambda > 0, \tag{3}$$
$$\Phi_{w,j}(x) = \Phi_{x,j}(w), \qquad \forall w, x \in \mathscr{V}. \tag{4}$$

For $x, y \in \mathscr{V}$ and $j \in [N]$, define

$$\mathscr{F}_{x,y,j} = \{w \in \mathscr{V} : \Phi_{w,j}(x) = \Phi_{w,j}(y)\}.$$

If $x \sim y$, then clearly $\mathscr{F}_{x,y,j} = \mathscr{V}$. For $x, y \in \mathscr{V}$ with $x \nsim y$ we want give a geometrical description of $\mathscr{F}_{x,y,j}$. Let $r_1 \in G$ be such that $\Phi_{w,j}(x) = \langle U_{r_1} w, x\rangle$ and $r_2 \in G$ such that $\Phi_{w,j}(y) = \langle U_{r_2} w, x\rangle$. Then $\langle w, U_{r_1^{-1}} x - U_{r_2^{-1}} y\rangle = 0$ which implies that

$$\mathscr{F}_{x,y,j} \subset \bigcup_{h_1,h_2 \in G} \{U_{h_1} x - U_{h_2} y\}^{\perp}. \tag{5}$$

On the other hand, each $\{U_{h_1} x - U_{h_2} y\}^{\perp} \subset \mathscr{V}$ is a proper hyperplane because $U_{h_1} x - U_{h_2} y \neq 0$ for any $h_1, h_2 \in G$ whenever $x \nsim y$. As a result, we conclude that $\mathscr{F}_{x,y,j}$ is contained within a finite union of $(d-1)$-dimensional hyperplanes.

For a filter bank $\mathbf{w} = (w_1, \ldots, w_p)$ and a set $S \subset [N] \times [p]$, we denote $\mathscr{F}_S$ as all collections of p-tuples $\mathbf{w} = (w_1, \ldots, w_p)$ such that the filter bank $\{\Phi_{i,j} : i \in [p], j \in [S_i]\}$ fails to separate all possible non-equivalent points $x, y \in \mathscr{V}$. This means that,

$$\mathscr{F}_S = \Big\{\mathbf{w} \in \mathscr{V}^p : \exists x, y \in \mathscr{V} \text{ with } x \nsim y \\ \text{and } \Phi_{i,j}(x) = \Phi_{i,j}(y), \forall i \in [p],\ \forall j \in S_i\Big\}.$$

Following the notation in [15], we will refer to the set $\mathscr{F}_S$ as the "bad set" because it contains the set of p-tuples $\mathbf{w} \in \mathscr{V}^p$ that fail to construct an injective embedding $\Phi_{\mathbf{w},S}$. We will establish requirements for the set S so that the "bad set" $\mathscr{F}_S$ is a subset of a Zariski-closed, proper subset of $\mathscr{V}^p$.

Let

$$\Gamma = \{(x, y) \in \mathscr{V}^2 : x \nsim y\} \tag{6}$$

be the set of all non-equivalent pairs of vectors. It's important to notice that Γ is an open set, with its complement being a finite union of closed linear subspaces of dimension $d = \dim(\mathscr{V})$. If the assumptions of Theorem 3.2 are satisfied, we can observe that

$$\begin{aligned}\mathscr{F}_A &\subset \bigcup_{(x,y)\in\Gamma} \bigcup_{h_1,\dots,h_{2p}\in G} \left(\{U_{h_1}x - U_{h_2}y\}^{\perp} \times \cdots \times \{U_{h_{2p-1}}x - U_{h_{2p}}y\}^{\perp}\right) \\ &= \bigcup_{h_1,\dots,h_{2p}=1}^{N} \bigcup_{(x,y)\in\Gamma} \left(\{U_{h_1}x - U_{h_2}y\}^{\perp} \times \cdots \times \{U_{h_{2p-1}}x - U_{h_{2p}}y\}^{\perp}\right).\end{aligned}$$

For fixed $\{h_1, \dots, h_{2p}\} \in G$, set

$$\mathscr{F}_{h_1,\dots,h_{2p}} = \bigcup_{(x,y)\in\Gamma} \{U_{h_1}x - U_{h_2}y\}^{\perp} \times \cdots \times \{U_{h_{2p-1}}x - U_{h_{2p}}y\}^{\perp}.$$

Notice that because G is a finite group in order to prove Theorem 3.1 is enough to show that the for any choice of $h_1, \dots, h_{2p} \in G$ the set $(\mathscr{F}_{h_1,\dots,h_{2p}})^c$ contains a Zariski open nonempty subset of $\mathscr{V}^p$.

Recall that the group G has size $N = |G|$. For fixed $2p$ elements $h_1, \dots, h_{2p} \in G$, we denote by $h_{g_1,\dots,g_{2p}} : \mathscr{V} \times \mathscr{V} \to \mathscr{V}^p$ the linear map

$$f_{h_1,\dots,h_{2p}}(x, y) = (U_{h_1}x - U_{h_2}y, \dots, U_{h_{2p-1}}x - U_{h_{2p}}y).$$

Observe that

$$\dim(\mathrm{Ran}(f_{h_1,\dots,h_{2p}})) = \dim(\mathscr{V} \times \mathscr{V}) \leq 2d - r$$

where $r = \dim(\ker(f_{g_1,\dots,g_{2p}}))$.

Next, let $C = f_{h_1,\dots,h_{2p}}(\Gamma)$ denote the image of set Γ through the linear map $f_{h_1,\dots,h_{2p}}$. Note that C is a semialgebraic subset of $\mathscr{V}^p$ of dimension $2d - r$.

Consider the set

$$B = C \cap S_1(\mathscr{V}^p).$$

We have already shown that C is an open subset of a $2d - r$-dimensional subspace of $\mathscr{V}^p$. Thus, B is an open subset of the $(2d - r - 1)$-dimensional unit sphere in $\mathscr{V}^p$, and hence a $(2d - r - 1)$-smooth manifold.

Moreover C is an semialgebraic set, so it is a semialgebraic set of dimension $(2d - r - 1)$.

Now, let $\mathbf{w} = (w_1, \dots, w_p) \in (\mathscr{V} \setminus \{0\})^p$. For easiness of notation we define $\{0 \oplus x \oplus 0\}_p^i \in \mathscr{V}^p$ to be the element of vector space $\mathscr{V}^p$, where in i-th entry is the vector x and all other $p-1$ entries are equal with the zero vector. Throughout the rest of this paper and for each such $\mathbf{w}$, we fix a choice of $p(d-1)$ vectors $h_1, \ldots, h_{p(d-1)} \in \mathscr{V}^p$ so that the set

$$\left\{\{0 \oplus \frac{w_1}{\|w_1\|} \oplus 0\}_p^1, \ldots, \{0 \oplus \frac{w_p}{\|w_p\|} \oplus 0\}_p^p, h_1, \ldots, h_{p(d-1)}\right\}$$

forms a basis in $\mathscr{V}^p$. The choice of h_i's need not change continuously with $\mathbf{w}$. Using Gram-Schmidt, we turn this set into an orthonormal basis in $\mathscr{V}^p$ of the form

$$\left\{\{0 \oplus \frac{w_1}{\|w_1\|} \oplus 0\}_p^1, \ldots, \{0 \oplus \frac{w_1}{\|w_1\|} \oplus 0\}_p^p, \mathbf{e}_1^{\mathbf{w}}, \ldots, \mathbf{e}_{p(d-1)}^{\mathbf{w}}\right\}.$$

Of course, the vectors $\mathbf{e}_1^{\mathbf{w}}, \ldots, \mathbf{e}_{p(d-1)}^{\mathbf{w}}$ depend on w as well as on the choices of the auxiliary $p(d-1)$ vectors $h_1, \ldots, h_{p(d-1)}$. However, we shall discard the implicit dependency on these auxiliary vectors h_i's from our notation.

For each $\mathbf{w} = (w_1, \ldots, w_p) \in (\mathscr{V} \setminus \{0\})^p$ there is a ball of radius $\rho_{\mathbf{w}} > 0$, $U_{\mathbf{w}} := B(\rho_{\mathbf{w}}, \mathbf{w}) \subset \mathscr{V}^p$ open in the ambient space centered at $\mathbf{w}$ such that for all $\mathbf{v} = (v_1, \ldots v_p) \in B(2\rho_{\mathbf{w}}, \mathbf{w})$ we have that the pd vectors

$$\left\{\{0 \oplus \frac{v_1}{\|v_1\|} \oplus 0\}_p^1, \ldots, \{0 \oplus \frac{v_p}{\|v_p\|} \oplus 0\}_p^p, \mathbf{e}_1^{\mathbf{w}}, \ldots, \mathbf{e}_{p(d-1)}^{\mathbf{w}}\right\}$$

still span the $\mathscr{V}^p$. Note that $\mathbf{e}_1^{\mathbf{w}}, \ldots, \mathbf{e}_{p(d-1)}^{\mathbf{w}}$ depend on $\mathbf{w}$ but are independent from $\mathbf{v}$ Using Gram-Schmidt process we transform this, non necessary orthonormal, basis, into the orthonormal basis

$$\left\{\{0 \oplus \frac{v_1}{\|v_1\|} \oplus 0\}_p^1, \ldots, \{0 \oplus \frac{v_p}{\|v_p\|} \oplus 0\}_p^p, \mathbf{e}_1^{\mathbf{w},\mathbf{v}}, \ldots, \mathbf{e}_{p(d-1)}^{\mathbf{w},\mathbf{v}}\right\}.$$

Note that each element of the orthonormal basis we constructed, depends continuously on $\mathbf{v}$.

For fixed $\mathbf{x} = (x_1, \ldots, x_p) \in \mathscr{V}^p$, denote by $F_{\mathbf{x}}$ the linear subspace

$$F_{\mathbf{x}} = \{\mathbf{y} = (y_1, \ldots, y_p) \in \mathscr{V}^p \ : \ \langle y_1, x_1\rangle = \cdots = \langle y_p, x_p\rangle = 0\}.$$

Note that for each $\mathbf{w} \in (\mathscr{V} \setminus \{0\})^p$ and $\mathbf{v} \in U_{\mathbf{w}}$, the orthonormal set $\mathbf{e}_1^{\mathbf{w},\mathbf{v}}, \ldots, \mathbf{e}_{p(d-1)}^{\mathbf{w},\mathbf{v}}$ is an orthonormal basis for the linear space $F_{\mathbf{v}}$.

Now for $M \subset (\mathscr{V} \setminus \{0\})^p$, let $E_M = \{(\mathbf{x}, \mathbf{y}) : \mathbf{x} \in M, \mathbf{y} \in F_{\mathbf{x}}\}$ denote a subset of $\mathscr{V}^{2p}$ and $\pi : E_M \to M$ be the projection on the first component, i.e. $\pi(\mathbf{x}, \mathbf{y}) = \mathbf{x}$.

Proposition 3.4. *Suppose that $M \subset (\mathscr{V} \setminus \{0\})^p$ is a k-dimensional algebraic variety. Then, (E_M, π, M) is an real analytic vector bundle with a k-dimensional base M, bundle projection π, $k + p(d-1)$-dimensional total space E_M, and linear fibers of dimension $p(d-1)$.*

Proof. For each $\mathbf{w} = (w_1, \ldots, w_p) \in M$, consider the map $\psi_{\mathbf{w}} : \pi^{-1}(U_{\mathbf{w}}) \to U_{\mathbf{w}} \times \mathbb{R}^{p(d-1)}$ defined by

$$\psi_{\mathbf{x}}(\mathbf{v}, z) = (\mathbf{v}, (\langle z, e_1^{\mathbf{w},\mathbf{v}} \rangle, \ldots, \langle z, e_{p(d-1)}^{\mathbf{w},\mathbf{v}} \rangle))$$

where $\mathbf{v} = (v_1, \ldots, v_p) \in U_{\mathbf{w}}$, and the map $\phi_{\mathbf{w}} : U_{\mathbf{w}} \times \mathbb{R}^{p(d-1)} \to \pi^{-1}(U_{\mathbf{w}})$ defined by

$$\phi_{\mathbf{w}}(\mathbf{v}, (c_1, \ldots, c_{p(d-1)})) = (\mathbf{v}, \sum_{i=1}^{p(d-1)} c_i e_i^{w,v}).$$

It is clear that $\phi_{\mathbf{w}} \circ \psi_{\mathbf{w}} = \mathrm{id}$ and $\psi_{\mathbf{w}} \circ \phi_{\mathbf{w}} = \mathrm{id}$ and hence both maps are bijections. Additionally, both $\phi_{\mathbf{w}}$ and $\psi_{\mathbf{w}}$ are continuous and, therefore, homeomorphisms. This shows that (E_M, π_M, M) is a topological vector bundle. □

Proposition 3.5. *Recall that $B = f_{g_1,\ldots,g_{2p}}(\Gamma) \subset (\mathscr{V} \setminus \{0\})^m$ is semialgebraic set of dimension $2d - r - 1$, where $r = \dim(\ker f_{g_1,\ldots,g_{2p}})$. There exists a finite collection of trivial vector bundles (E_j, π_j, B_j) with base manifolds B_j of same dimension, bundle projections π_j, total spaces $E_j = E_{B_j}$ (compatible with the definition E_M introduced earlier), and linear fibers of dimension $m(d-1)$ such that $\bigcup_j C_j = C$ and $\bigcup_j E_j = E_B$. Thus, (E_j, π_j, B_j) provide a finite cover for the vector bundle (E_B, π, C).*

Proof. We want to find a finite cover, $\{B_j\}_{j=1}^L$, of B so that each (E_j, π_j, B_j) is a trivial vector bundle.

The product of unit spheres $S_1(\mathscr{V})^p$ is compact, and hence we can find a finite collection $\{\mathbf{w}_i\}_{i=1}^L$, $\mathbf{w}_i \in S_1(\mathscr{V})^p$ such that $\{U_{\mathbf{w}_i}\}_{i=1}^L$ is a cover of $S_1(\mathscr{V})^p$, where each $U_{\mathbf{w}}$ is some ball centred at $\mathbf{w}$. Next, define

$$\tilde{U}_{\mathbf{w}} = \{\mathbf{v} = (v_1, \ldots, v_p) \in (\mathscr{V} \setminus \{0\})^p : (\frac{v_1}{\|v_1\|}, \ldots, \frac{v_p}{\|v_p\|}) \in U_{\mathbf{w}}\},$$

and note that the sets $B_i = \tilde{U}_{\mathbf{w}_i} \cap B$, for $i \in [L]$, form a finite cover of B.

Now, we will show that the triple (E_j, π, B_j) is a trivial vector bundle. For this, we have to find $p(d-1)$ independent global sections. For any $\mathbf{v} =$

$(v_1, \ldots, v_p) \in B_j$, recall that the following set of vectors forms an orthonormal basis:

$$\left\{ \{0 \oplus \frac{v_1}{\|v_1\|} \oplus 0\}_p^1, \ldots, \{0 \oplus \frac{v_p}{\|v_p\|} \oplus 0\}_p^p, \mathbf{e}_1^{\mathbf{w},\mathbf{v}}, \ldots, \mathbf{e}_{p(d-1)}^{\mathbf{w},\mathbf{v}} \right\}.$$

Now, if we define the maps $s_{l,j} : B_j \to E_j$ by

$$s_l(\mathbf{v}) = (\mathbf{v}, e_l^{\mathbf{w}_j,\mathbf{v}}),$$

it is clear that $\{s_{l,j}\}_{l=1}^{p(d-1)}$ forms a set of $p(d-1)$ independent global sections in B_j. We conclude that (B_j, π, E_j) is a trivial vector bundle. $\square$

Now we can complete the proof of Theorem 3.2.

Proof of Theorem 3.2. Now, define the map $\mathrm{P}_j : B_j \times \mathbb{R}^{p(d-1)} \to \mathscr{V}^p$ by

$$\mathrm{P}_j(\mathbf{v}, \mathbf{c}) = \sum_{i=1}^{p(d-1)} c_i e_i^{\mathbf{w}^j,\mathbf{v}}.$$

We have already shown that for a fixed $\mathbf{w} \in \mathscr{V}^p$ the mapping $\mathbf{v} \mapsto e^{\mathbf{w},\mathbf{v}}$ is semialgebraic, hence P_j is also morphism as a linear combination of semialgebraic maps. Observe that

$$\bigcup_j \mathrm{P}_j(B_j \times \mathbb{R}^{p(d-1)}) = \mathscr{F}_{g_1,\ldots,g_{2p}}.$$

Notice that B_j is an semialgebraic set as a intersection of two semialgebraic sets, so $B_j \times \mathbb{R}^{p(d-1)}$ is a semialgebraic set of dimension

$$2d - r - 1 + p(d-1) \leq 2d - 1 + p(d-1)$$

and also that

$$p \geq 2d \implies 2d - 1 + p(d-1) < pd.$$

For every j, P_j is semialgebraic, and $B_j \times \mathbb{R}^{p(d-1)}$ is a semialgebraic set of dimension at most $2d-1+p(d-1)$, so from theorem 1.12 $\mathrm{P}_j(B_j \times \mathbb{R}^{p(d-1)})$ is a semialgebraic set of dimension at most $2d-1+p(d-1)$ and from corollary 1.14 it is a nowhere dense set with zero Lebesgue measure. $\square$

3.1 Coorbit embedding

Up to this point, we have focused on the scenario where we used only one element from each column of the matrix S for the construction of the embedding $\Phi_{\mathbf{w},S}$. Now, we aim to explore the situation where we are permitted to use more than one element from each column.

We will demonstrate that in this case, one can find p smaller than $2d$ such that for almost every $\mathbf{w} \in \mathscr{V}^p$, the mapping $\Phi_{\mathbf{w},S}$ is injective in the quotient space $\hat{\mathscr{V}}$.

Theorem 3.6. *Let G be a finite group acting unitarily on $\mathscr{V} \cong \mathbb{R}^d$. For $1 \leq n \leq N-1$, let γ_n be the n-th entry of the sorted in decreasing order vector*

$$\gamma = \downarrow \{\min_{\lambda \in \mathrm{Sp}(g)} \mathrm{rank}[g - \lambda\, \mathrm{I}]\}_{g \neq \mathrm{I}_d}$$

where

$$\mathrm{Sp}(g) = \{\lambda \in \mathbb{R} : \det(U_g - \lambda\, \mathrm{I}) = 0\}$$

and

$$p_n = 2d - \gamma_{N-n+1}.$$

Notice that $p_n \geq d+1$. Choose an integer p such that $p_n \leq p \leq 2d$ and a set $S \subset [p] \times [N]$ such that $|S_k| = n$ for $1 \leq i \leq 2d-p$ and $|S_i| = 1$ for $2d-p+1 \leq i \leq p$. Note that S has cardinality of $m = (2d-p)n + 2p - 2d$. Then, for a generic with respect to Zariski topology, $\boldsymbol{w} \in \mathscr{V}^p$, the map $\Phi_{\boldsymbol{w},S}$ is injective, i.e. for all $x, y \in \mathscr{V}$ it holds

$$\Phi_{\boldsymbol{w},S}(x) = \Phi_{\boldsymbol{w},S}(y) \iff x \sim y.$$

To prove Theorem 3.6, we will employ a procedure similar to the one used for Theorem 3.2, and thus, our notation will also be analogous.

Recall that

$$\mathscr{F}_S = \{\mathbf{w} \in \mathscr{V}^p : \exists (x,y) \in \Gamma \text{ such that } \Phi_{i,j}(x) = \Phi_{i,j}(y)\ \forall (i,j) \in S\}$$

where Γ has been defined in (6).

To establish the proof of Theorem 3.6, it suffices to demonstrate that for every S satisfying the assumptions of the theorem, the set $(\mathscr{F}_S)^c$ contains a Zariski open nonempty of $\mathscr{V}^p$.

For fixed $r \in [N]$ we define the set of group elements

$$H_r^* = \{g_{i_1}, \ldots, g_{i_r} \in G^r : 1 \leq i_1 < \cdots < i_r \leq N\}.$$

Notice that

$$\begin{aligned}
\mathscr{F}_S &= \{\mathbf{w} \in \mathscr{V}^p : \exists x, y \in \mathscr{V},\ x \nsim y : \Phi_{i,j}(x) = \Phi_{i,j}(y),\ \forall (i,j) \in S\} \\
&\subset \bigcup_{(x,y)\in\Gamma} \bigcup_{\substack{\pi_1,\ldots,\pi_p \in S_N \\ \sigma_1,\ldots,\sigma_p \in S_N}} \{\mathbf{w} \in \mathscr{V}^p : \langle x, U_{g_{\pi_i(j)}} w_i\rangle = \langle y, U_{g_{\sigma_i(j)}} w_i\rangle,\ \forall (i,j) \in S\} \\
&= \bigcup_{\substack{\pi_1,\ldots,\pi_p \in S_N \\ \sigma_1,\ldots,\sigma_p \in S_N}} \bigcup_{(x,y)\in\Gamma} \bigotimes_{k=1}^{p} \{\mathbf{w} \in \mathscr{V}^p : \langle U^{-1}_{g_{\pi_i(j)}} x - U_{g_{\sigma_i(j)}} y, w_i\rangle = 0,\ \forall (i,j) \in S\} \\
&= \bigcup_{a_i, b_i \in H^*_{m_i}} \bigcup_{(x,y)\in\Gamma} \bigotimes_{i=1}^{p} \Big(\bigcap_{j=1}^{m_i} \{U_{a_i(j)} x - U_{b_i(j)} y\}^{\perp} \Big).
\end{aligned}$$

For fixed $a_i, b_i \in H^*_{m_i}$, $i \in [p]$, we introduce the set

$$\mathscr{F}_{\substack{a_1,\ldots,a_p \\ b_1,\ldots,b_p}} = \bigcup_{(x,y)\in\Gamma} \Big(\bigotimes_{i=1}^{p} \bigcap_{j=1}^{m_i} \{U_{a_i(j)} x - U_{b_i(j)} y\}^{\perp} \Big).$$

Notice that because group G is finite, it suffices to show that for any choice of $a_i, b_i \in H^*_{m_i}$, $i \in [p]$ where m_i and p satisfy the requirements of Theorem 3.1, the set $(\mathscr{F}_{\substack{a_1,\ldots,a_p \\ b_1,\ldots,b_p}})^c$ contains a Zariski open, subset of $\mathscr{V}^p$.

Definition 3.7. *For fixed $r \in [N]$, $q \in [r-1]$ and $a, b \in H_r^*$, we define the following set:*

$$\Gamma_q^{a,b} = \{(x,y) : \dim(\mathrm{span}(U_{a(1)}x - U_{b(1)}y, \ldots, U_{a(m_i)}x - U_{b(m_i)}y)^{\perp}) \geq d - r + q\}.$$

Furthermore, for $q \in [r-2]$, let

$$\begin{aligned}
\Delta_q^{a,b} &= \Gamma_q^{a,b} \setminus \Gamma_{q+1}^{a,b} = \\
&= \{(x,y) \in \mathscr{V}^2 : \dim(\mathrm{span}(U_{a(1)}x - U_{a(1)}y, \ldots, U_{a(r)}x - U_{a(r)}y)^{\perp}) = d - r + q\}.
\end{aligned}$$

Also, for $a_i, b_i \in H^*_{m_i}$, $i \in [p]$ and $x, y \in \mathscr{V}$, let

$$q_i = m_i - \dim(\mathrm{span}(U_{a_i(1)}x - U_{b_i(1)}y, \ldots, U_{a_i(m_i)}x - U_{b_i(m_i)}y)).$$

Notice that

$$\Gamma \subset \bigcup_{\substack{q_1,\ldots,q_p\\ q_i\in[m_i-1]}} \bigcap_{k=1}^{p} \Delta_{q_k}^{a_k,b_k}.$$

Therefore,

$$\mathscr{F}_{\substack{a_1,\ldots,a_p\\ b_1,\ldots,b_p}} \subset \bigcup_{\substack{q_1,\ldots,q_p\\ q_i\leq m_i-1}} \bigcup_{(x,y)\in\bigcap_{k=1}^{p}\Delta_{q_k}^{a_k,b_k}} \Big(\bigotimes_{i=1}^{p} \operatorname{span}(U_{a_i(1)}x - U_{b_i(1)}y,\ldots,U_{a_i(m_i)}x - U_{b_i(m_i)}y)^{\perp}\Big).$$

Recall that Theorem 3.6 assumes that $m_1 = \cdots = m_q = n$ and $m_{q+1} = \cdots = m_p = 1$. Let

$$\mathscr{F}^1_{\substack{a_1,\ldots,a_p\\ b_1,\ldots,b_p}} = \bigcup_{(x,y)\in\bigcap_{k=1}^{p}\Delta_{m_k-1}^{a_k,b_k}} \Big(\bigotimes_{i=1}^{p} \operatorname{span}(U_{a_i(1)}x - U_{b_i(1)}y,\ldots,U_{a_i(m_i)}x - U_{b_i(m_i)}y)^{\perp}\Big)$$

and

$$\mathscr{F}^2_{\substack{a_1,\ldots,a_p\\ b_1,\ldots,b_p}} = \bigcup_{\substack{q_1,\ldots,q_p\\ q_i\leq m_i-2, i\in[q]}} \bigcup_{(x,y)\in\bigcap_{k=1}^{p}\Delta_{q_k}^{a_k,b_k}} \Big(\bigotimes_{i=1}^{p} \operatorname{span}(U_{a_i(1)}x - U_{b_i(1)}y,\ldots,U_{a_i(m_i)}x - U_{b_i(m_i)}y)^{\perp}\Big)$$

Notice that for $a, b \in H_n^*$

$$\begin{aligned}\Gamma^{a,b}_{n-1} = \Delta^{a,b}_{n-1} = \{(x,y)\in\mathscr{V}^2 : \exists \boldsymbol{c} = (c_1,\ldots,c_{n-1})\in\mathbb{R}^{n-1}\\ : (U_{a(1)}x - U_{b(1)}y - c_i(U_{a(i+1)}x - U_{b(i+1)}y) = 0,\ \forall i\in[n-1]\}.\end{aligned}$$

We define $\Lambda_i = (\lambda_{i-1},\lambda_i)$ for $i \in [k+1]$, where by a slightly abuse of notation we let $\lambda_0 = -\infty$ and $\lambda_{k+1} = +\infty$. For fixed $a, b \in H_n^*$ let the map $\ell_{a,b} : \mathbb{R}^{n-1}\times\mathscr{V}^2 \to \mathscr{V}^{n-1}$ defined by,

$$\begin{aligned}\ell_{a,b}(c_1,\ldots,c_{n-1},x,y) =& U_{a(1)}x - U_{b(1)}y - c_1(U_{a(2)}x - U_{b(2)}y),\\ &\ldots, U_{a(1)}x - U_{b(1)}y - c_{n-1}(U_{a(n)}x - U_{b(n)}y).\end{aligned}$$

We also define the following auxiliary set:

$$\Gamma^1_{\substack{a_1,\ldots,a_p\\ b_1,\ldots,b_p}} = \{(\boldsymbol{C},x,y)\in\mathbb{R}^{q\times(n-1)}\times\Gamma : \ell_{a_i,b_i}(\boldsymbol{C_i},x,y) = 0,\ \forall i\in[q]\}.$$

Notice that

$$\mathscr{F}^1_{\substack{a_1,\dots,a_p\\ b_1,\dots,b_p}} \subset \bigcup_{(\boldsymbol{C},x,y)\in\Gamma^1_{\substack{a_1,\dots,a_p\\ b_1,\dots,b_p}}} \{U_{a_1(1)}x - U_{b_1(1)}y\}^\perp \times \cdots \times \{U_{a_p(1)}x - U_{b_p(1)}y\}^\perp$$

and

$$\mathscr{F}^2_{\substack{a_1,\dots,a_p\\ b_1,\dots,b_p}} \subset \bigcup_{\substack{q_1,\dots,q_p\\ q_i\le m_i-2, i\in[q]}} \bigcup_{(x,y)\in\cap_{k=1}^p \Delta^{a_k,b_k}_{q_k}} \Big(\bigotimes_{i=1}^p \operatorname{span}(U_{a_i(1)}x - U_{b_i(1)}y, \dots, U_{a_i(m_i)}x - U_{b_i(m_i)}y)^\perp\Big).$$

Now we will show some helpful lemmas before showing that $\mathscr{F}^2_{\substack{a_1,\dots,a_p\\ b_1,\dots,b_p}}$ is a zero measure subset of $\mathscr{V}^p$.

For fixed $h_1,\dots,h_{2m}\in G$, let $f_{h_1,\dots,h_{2m}}:\mathscr{V}\times\mathscr{V}\to\mathscr{V}^m$ denote the linear map

$$f_{h_1,\dots,h_{2m}}(x,y) = (U_{h_1}x - U_{h_2}y, \dots, U_{h_{2m-1}}x - U_{h_{2m}}y).$$

Now, let

$$C_2 = f_{h_1,\dots,h_{2m}}\Big(\bigcup_{\substack{q_1,\dots,q_p\\ q_i\le m_i-2, i\in[q]}} \bigcap_{k=1}^p \Delta^{a_k,b_k}_{q_k}\Big)$$

denote the image of the open set

$$\bigcup_{\substack{q_1,\dots,q_p\\ q_i\le m_i-2, i\in[q]}} \bigcap_{k=1}^p \Delta^{a_k,b_k}_{q_k} \subset \mathscr{V}^2$$

through the linear map $f_{h_1,\dots,h_{2m}}$. Note C_2 is a semialgebraic set of dimension $2d-r\le 2d$.

Following the notation of previous section we define the set,

$$B_2 = C_2 \cap S^1(\mathscr{V}^m).$$

We have already shown that C_2 is a semialgebraic of dimension $2d-r\le 2d$, therefore B_2 is a open subset of a $(2d-r-1)$-dimensional unit sphere in $\mathscr{V}^{2d}$ and hence a semialgebraic set in $\mathscr{V}^m$ of dimension $2d-r-1\le 2d-1$.

For fixed $\mathbf{w}=(w_1,\dots,w_m)\in B_2$ notice that for every $k\in[q]$, exists $(k-1)n+1\le i_k, j_k\le kn$ such that w_{i_k} and w_{j_k} are linearly independent vectors. After perform permutation on elements $\{w_{(k-1)n+1}, \dots, w_{kn},\ k \in [q]\}$ we can always assume that $i_k = (k-1)n+1$ and $j_k = (k-1)n+2$ Also, for any pair (w_{i_k}, w_{j_k}), let $(e(w_{i_k}), e(w_{j_k}))$ be the corresponding vectors after we perform the Gram-Schmidt process to the pair (w_{i_k}, w_{j_k}). Notice that we can choose vectors $f_1, \dots, f_{d(p-2)} \in \mathscr{V}$ so that

$$\Big\{\{0 \oplus w_1 \oplus 0\}_p^1, \{0 \oplus w_2 \oplus 0\}_p^1, \{0 \oplus w_{n+1} \oplus 0\}_p^2, \\ \{0 \oplus w_{n+2} \oplus 0\}_p^2, \dots, \{0 \oplus w_{(q-1)n+1} \oplus 0\}_p^q, \{0 \oplus w_{(q-1)n+2} \oplus 0\}_p^q, \\ \{0 \oplus w_{nq+1} \oplus 0\}_p^{q+1}, \dots, \{0 \oplus w_m \oplus 0\}_p^p, f_1, \dots, f_{d(p-2)}\Big\}$$

forms a basis in $\mathscr{V}^p$. Use Gram-Schmidt to turn this set into an orthonormal basis in $\mathscr{V}^p$ of the form

$$\Big\{\{0 \oplus e(w_1) \oplus 0\}_p^1, \{0 \oplus e(w_2) \oplus 0\}_p^1, \{0 \oplus e(w_{n+1}) \oplus 0\}_p^2, \\ \{0 \oplus e(w_{n+2}) \oplus 0\}_p^2, \dots, \{0 \oplus e(w_{(q-1)n+1}) \oplus 0\}_p^q, \dots, \{0 \oplus e(w_{(q-1)n+2}) \oplus 0\}_p^q, \\ \{0 \oplus \frac{w_{nq+1}}{\|w_{nq+1}\|} \oplus 0\}_p^{q+1}, \dots, \{0 \oplus \frac{w_m}{\|w_m\|} \oplus 0\}_p^p, h_1, \dots, h_{d(p-2)}\Big\}$$

For all $\mathbf{v} = (v_1, \dots v_m) \in B(2\rho_w, w)$ we have that the pd vectors

$$\Big\{\{0 \oplus e(v_1) \oplus 0\}_p^1, \{0 \oplus e(v_2) \oplus 0\}_p^1, \{0 \oplus e(v_{n+1}) \oplus 0\}_p^2, \\ \{0 \oplus e(v_{n+2}) \oplus 0\}_p^2, \dots, \{0 \oplus e(v_{(q-1)n+1}) \oplus 0\}_p^q, \{0 \oplus e(v_{(q-1)n+2}) \oplus 0\}_p^q, \\ \{0 \oplus \frac{v_{nq+1}}{\|v_{nq+1}\|} \oplus 0\}_p^{q+1}, \dots, \{0 \oplus \frac{v_m}{\|v_m\|} \oplus 0\}_p^p, h_1, \dots, h_{d(p-2)}\Big\}$$

still span the $\mathscr{V}^p$. Using Gram-Schmidt process we transform this basis into an orthonormal one:

$$\Big\{\{0 \oplus e(v_1) \oplus 0\}_p^1, \{0 \oplus e(v_2) \oplus 0\}_p^1, \{0 \oplus e(v_{n+1}) \oplus 0\}_p^2, \\ \{0 \oplus e(v_{n+2}) \oplus 0\}_p^2, \dots, \{0 \oplus e(v_{(q-1)n+1}) \oplus 0\}_p^q, \{0 \oplus e(v_{(q-1)n+2}) \oplus 0\}_p^q, \\ \{0 \oplus \frac{v_{nq+1}}{\|v_{nq+1}\|} \oplus 0\}_p^{q+1}, \dots, \{0 \oplus \frac{v_m}{\|v_m\|} \oplus 0\}_p^p, \mathbf{e}_1^{\mathbf{w},\mathbf{v}}, \dots, \mathbf{e}_{d(p-2)}^{\mathbf{w},\mathbf{v}}\Big\}$$

Following the notation of Theorem 3.2 for each $\mathbf{x} = (x_1, \dots, x_m) \in \mathscr{V}^m$, we denote by $F_{\mathbf{x}}$ the linear subspace $F_{\mathbf{x}} = \{x_1\}^\perp \cap \cdots \cap \{x_n\}^\perp \times \cdots \times \{x_{2q-n+1}\}^\perp \cap$

$\cdots \cap \{x_{2q}\}^\perp \times \{x_{2q+1}\}^\perp \times \cdots \times \{x_m\}^\perp \subset \mathscr{V}^p$. Note that for each $\mathbf{w} \in B_2$ and $\mathbf{v} \in U_\mathbf{w}$, the orthonormal set $\mathbf{e}_1^{\mathbf{w},\mathbf{v}}, \ldots, \mathbf{e}_{d(p-2)}^{\mathbf{w},\mathbf{v}}$ is an orthonormal basis for the linear space $F_\mathbf{v}$. Finally, for any subset M of B_2, let $E_M = \{(\mathbf{x}, \mathbf{y}) : \mathbf{x} \in M, \mathbf{y} \in F_\mathbf{x}\} \subset \mathscr{V}^{2d+p}$ and $\pi : E_M \to M$ be the projection on first component, that is $\pi(\mathbf{x}, \mathbf{y}) = \mathbf{x}$.

Proposition 3.8. *Suppose that $M \subset B_2$ is a l-dimensional manifold. Then (E_M, π, M) is vector bundle, with l-dimensional base M, bundle projection π, $l + d(p-2)$-dimensional total space E_M, and linear fibers of dimension $d(p-2)$.*

For each $\mathbf{w} = (w_1, \ldots, w_m) \in M$, let $\psi_\mathbf{w} : \pi^{-1}(U_\mathbf{w}) \to U_\mathbf{w} \times \mathbb{R}^{d(p-2)}$, where,

$$\psi_\mathbf{w}(\mathbf{v}, \mathbf{z}) = (\mathbf{v}, (\langle \mathbf{z}, \mathbf{e}_1^{\mathbf{w},\mathbf{v}} \rangle, \ldots, \langle \mathbf{z}, \mathbf{e}_{d(p-2)}^{\mathbf{w},\mathbf{v}} \rangle))$$

and $\phi_\mathbf{w} : U_\mathbf{w} \times \mathbb{R}^{d(p-2)} \to \pi^{-1}(U_\mathbf{w})$ where

$$\phi_\mathbf{w}(\mathbf{v}, (c_1, \ldots, c_{d(p-2)})) = (\mathbf{v}, \sum_{i=1}^{d(p-2)} c_i \mathbf{e}_i^{\mathbf{w},\mathbf{v}})$$

It is clear that $\phi_\mathbf{w}$ and $\psi_\mathbf{w}$ are inverse to each other and hence they both are bijections. Furthermore, both $\phi_\mathbf{w}$ and $\psi_\mathbf{w}$ are continuous. Hence $\phi_\mathbf{w}$ and $\psi_\mathbf{w}$ are homeomorphisms. This proves that (E_M, π_M, M) is a topological vector bundle.

Proposition 3.9. *There exists a finite collection of trivial vector bundles $(E_{2,j}, \pi_j, B_{2,j})$, with base manifolds $B_{2,j}$, bundle projections π_j, total spaces $E_{2,j} = E_{B_{2,j}}$ (compatible with the definition E_M introduced earlier), and linear fibers of dimension $d(p-2)$, such that, $\bigcup_j B_{2,j} = B_2$ and $\bigcup_j E_{2,j} = E_{2,j}$. They provide a finite cover of the vector bundle (E_2, π, B_2).*

Proof. We want to show that we can find a finite cover of B_2, $\{B_{2,j}\}_{j=1}^L$, such that each $(E_j, \pi_{2,j}, B_{2,j})$ is a trivial vector bundle. Note that the set

$$D = \{(x_1, \ldots, x_m) \in \mathscr{V}^m \text{ such that } \|x_i\| = 1,\ \forall i \in [m] \\ \text{and } \langle x_{n(\kappa-1)}, x_{n(\kappa-1)+1} \rangle = 0,\ \text{for every } k \in [q]\}$$

is compact, hence we can find a finite collection $\{\mathbf{w}_i\}_{i=1}^K \in D$, such that $\{U_{\mathbf{w}_i}\}_{i=1}^K$ is a cover of D.

We also define the sets

$$Z = \{x_1, \dots, x_m \in \mathscr{V} : x_{n(\kappa-1)} \neq \lambda x_{n(\kappa-1)+1}, \forall k \in [q], \forall \lambda \in \mathbb{R}\}$$

and

$$\begin{aligned}\tilde{U}_{\mathbf{w}} =& \{(u_1, \dots, u_m) \in Z : \\ & (e(u_1), e(u_2), u_3, \dots, u_n, e(u_{n+1}), e(u_{n+2}), u_{n+3}, \dots, u_m) \in D\}\end{aligned}$$

i.e. $\tilde{U}_{\mathbf{w}}$ contains all $(v_1, \dots, v_m) \in Z$ such that if we replace $v_{n(\kappa-1)}$ and $v_{n(\kappa-1)+1}$ with $e(v_{n(\kappa-1)})$ and $e(v_{n(\kappa-1)+1})$ respectively, the transformed vector belongs in D.

Note that the sets $B_{2,i} = \tilde{U}_{\mathbf{w}_i} \cap B_2$, $1 \leq i \leq K$, collectively form a finite cover of B_2.

To demonstrate that the triple $(E2, j, \pi_j, B2, j)$ is a trivial vector bundle, it suffices to find $d(p-2)$ independent sections. For any $\mathbf{v} = (v_1, \dots, v_{2m}) \in B_{2,j}$, recall that the following set of vectors forms an orthonormal basis:

$$\begin{aligned}\Big\{&\{0 \oplus e(v_1) \oplus 0\}_p^1, \{0 \oplus e(v_2) \oplus 0\}_p^1, \{0 \oplus e(v_{(n+1)}) \oplus 0\}_p^2, \\ &\{0 \oplus e(v_{n+2}) \oplus 0\}_p^2, \dots, \{0 \oplus e(v_{(q-1)n+1}) \oplus 0\}_p^q, \dots, \{0 \oplus e(v_{(q-1)n+2}) \oplus 0\}_p^q, \\ &\{0 \oplus \frac{v_{nq+1}}{\|v_{nq+1}\|} \oplus 0\}_p^{nq+1}, \dots, \{0 \oplus \frac{v_m}{\|v_m\|} \oplus 0\}_p^p, \mathbf{e}_1^{\mathbf{w},\mathbf{v}}, \dots, \mathbf{e}_{d(p-2)}^{\mathbf{w},\mathbf{v}}\Big\}.\end{aligned}$$

Now let $s_l : B_{2,j} \to E_j$, be defined by

$$s_l(\mathbf{v}) = (\mathbf{v}, e_l^{\mathbf{w}_j,\mathbf{v}}).$$

Then $\{s_l\}_{l=1}^{d(p-2)}$ form a set of $d(p-2)$ independent global section in $C_{2,j}$, so $(B_{2,j}, \pi_j, E_j)$ is a trivial vector bundle. ∎

Proposition 3.10. *For any $a_i, b_i \in H^*_{m_i}$, $i \in [p]$. $\mathscr{F}^2_{\substack{a_1,\dots,a_p \\ b_1,\dots,b_p}}$ is a nowhere dense set with zero lebesgue measure.*

Proof. Let $\mathrm{P}_{2,j} : B_{2,j} \times \mathbb{R}^{d(p-2)} \to \mathscr{V}^p$ where,

$$\mathrm{P}_{2,j}(\mathbf{v}, \mathbf{c}) = \sum_{i=1}^{d(p-2)} c_i e_i^{\mathbf{w}_j,\mathbf{v}}.$$

We have already shown that $\mathrm{P}_{2,j}$ is semialgebraic map as a linear combination of semialgebraic maps. We notice that

$$\bigcup_j \mathrm{P}_{2,j}(B_{2,j} \times \mathbb{R}^{d(p-2)}) \supset \mathscr{F}^2_{\substack{a_1,\dots,a_p\\ b_1,\dots,b_p}}.$$

Because $\mathrm{P}_{2,j}$ is semialgebraic and for every j, $B_{2,j} \times \mathbb{R}^{p(d-1)}$ is a semialgebraic set of dimension $2d-1-r+p(d-1)$ from Theorem 1.12 $\mathrm{P}_{2,j}(B_{2,j} \times \mathbb{R}^{p(d-1)})$ is a semialgebraic set of dimension $\leq 2d-1-r+p(d-1)$ and from Corollary 1.14 it is a nowhere dense set with zero Lebesgue measure. □

Now, we still need to estimate the algebraic dimension of $\mathscr{F}^1_{\substack{a_1,\dots,a_p\\ b_1,\dots,b_p}}$.

Lemma 3.11. *For fixed $a_i, b_i \in H^*_{m_i}$, the set $\Gamma^1_{\substack{a_1,\dots,a_p\\ b_1,\dots,b_p}}$ is a closed subset of $\mathbb{R}^{q\times(n-1)} \times \mathscr{V}^2$.*

Proof. Let $\{(\boldsymbol{C}_n, x_n, y_n)\} \to (\boldsymbol{C}, x, y)$ be a convergence sequence in $\Gamma^1_{\substack{a_1,\dots,a_p\\ b_1,\dots,b_p}}$. In order to prove our lemma we need to show that $(\boldsymbol{C}, x, y)$ is an element of $\Gamma^1_{\substack{a_1,\dots,a_p\\ b_1,\dots,b_p}}$. Because $(\boldsymbol{C}_n, x_n, y_n) \in \Gamma^1_{\substack{a_1,\dots,a_p\\ b_1,\dots,b_p}}$

$$\ell_{a_i,b_i}((\boldsymbol{C}_n)_i, x_n, y_n) = 0 \ , \forall i \in [q], \forall n \in \mathbb{N}.$$

But ℓ_{a_i,b_i} is continuous function so $\ell_{a_i,b_i}(\boldsymbol{C}, x, y) = 0$, therefore

$$(\boldsymbol{C}, x, y) \in \Gamma^1_{\substack{a_1,\dots,a_p\\ b_1,\dots,b_p}}.$$

□

Lemma 3.12. *For fixed group elements $h_1, \dots, h_{2p} \in G$, let the map $f_{h_1,\dots,h_{2p}} : \mathbb{R}^{q\times(n-1)} \times \mathscr{V}^2 \to \mathscr{V}^p$, defined by $f_{h_1,\dots,h_{2p}}(\boldsymbol{C}, x, y) = U_{h_1}x - U_{h_2}y, \dots, U_{h_{2p-1}}x - U_{h_{2p}}y$. Then, the set $f_{h_1,\dots,h_{2p}}(\Gamma^1_{\substack{a_1,\dots,a_p\\ b_1,\dots,b_p}})$ is a semialgebraic set of dimension at most $2d - \gamma_n$.*

In order to prove Lemma 3.12 we to create a suitable partition of the set $\Gamma^1_{\substack{a_1,\dots,a_p\\ b_1,\dots,b_p}}$.

Note that the set $\Gamma^1_{\substack{a_1,\dots,a_p\\ b_1,\dots,b_p}}$ can be expressed as the disjoint union of the following auxiliary sets.

$$\Gamma^3_{\substack{a_1,\dots,a_p\\ b_1,\dots,b_p}} = \Gamma^1_{\substack{a_1,\dots,a_p\\ b_1,\dots,b_p}} \cap (E_G^{q\times(n-1)} \times \mathscr{V}^2)$$

and

$$\Gamma^4_{\substack{a_1,\dots,a_p\\b_1,\dots,b_p}} = \Gamma^1_{\substack{a_1,\dots,a_p\\b_1,\dots,b_p}} \setminus \Gamma^3_{\substack{a_1,\dots,a_p\\b_1,\dots,b_p}}.$$

Proposition 3.13. $\Gamma^4_{\substack{a_1,\dots,a_p\\b_1,\dots,b_p}}$ *is a semialgebraic set of dimension* $(q(n-1)+d)$.

Proof. Recall that

$$\begin{aligned}\Gamma^4_{\substack{a_1,\dots,a_p\\b_1,\dots,b_p}} &\subset \mathbb{R}^{(q-1)\times(n-1)} \times (\ker(\ell_{a_1,b_1}) \cap (\mathbb{R}^{n-2} \times E^c_G \times \mathscr{V}^2))\\ &= \mathbb{R}^{(q-1)\times(n-1)} \times \bigcup_{i\in[K+1]} (\ker(\ell_{a_1,b_1}) \cap (\mathbb{R}^{n-2} \times \Lambda_i \times \mathscr{V}^2)).\end{aligned}$$

Notice that

$$\begin{aligned}&\ker(\ell_{a_1,b_1}) \cap (\mathbb{R}^{n-2} \times \Lambda_i \times \mathscr{V}^2) \subset\\ &\subset \{(\boldsymbol{\lambda}, \lambda, x, (U_{h_1} - \lambda U_{h_2})^{-1}(U_{h_3} - \lambda U_{h_4})x) : \lambda \in \Lambda_i, x, y \in \mathscr{V},\ \boldsymbol{\lambda} \in \mathbb{R}^{n-2}\} \cong\\ &\cong \mathbb{R}^{n-2} \times \{(\lambda, x, (U_{h_1} - \lambda U_{h_2})^{-1}(U_{h_3} - \lambda U_{h_4})x) : \lambda \in \Lambda_i, x, y \in \mathscr{V}\}.\end{aligned}$$

Therefore is enough to show that the set

$$\{(\lambda, x, (U_{h_1} - \lambda U_{h_2})^{-1}(U_{h_3} - \lambda U_{h_4})x) : \lambda \in \Lambda_i, x, y \in \mathscr{V}\}$$

is a semialgebraic set of dimension at most $d + 1$.

Let,

$$E_i = \{(\lambda, x, (U_{h_1} - \lambda U_{h_2})^{-1}(U_{h_3} - \lambda U_{h_4})x) : \lambda \in \Lambda_i, x, y \in \mathscr{V}\}$$

and denote by $\pi_i : E_i \to \Lambda_i$ the projection $\pi(\lambda, x, y) = \lambda$. Also, consider the functions $\phi^i_{\mathbf{h}} : \Lambda_i\times\mathbb{R}^d \to \ker(\ell_{\mathbf{h}})\cap\Lambda_i\times\mathscr{V}^2$ and $\psi^i_{\mathbf{h}} : \ker(\ell_{\mathbf{h}})\cap\Lambda_i\times\mathscr{V}^2 \to \Lambda_i\times\mathbb{R}^d$ given respectively by

$$\phi^i_{\mathbf{h}}(\lambda, x) = (\lambda, x, (U_{h_1} - \lambda U_{h_2})^{-1}(U_{h_3} - \lambda U_{h_4})x)$$

and

$$\psi^i_{a,b}(\lambda, x, (U_{h_1} - \lambda U_{h_2})^{-1}(U_{h_3} - \lambda U_{h_4})x) = (\lambda, x).$$

Clearly, $\phi^i_{a,b}$ and $\psi^i_{a,b}$ are inverses to one another, are both continuous, and thus they are homeomorphisms. This tells us that (E_i, π_i, Λ_i) is a topological vector bundle.

Additionally, the map $\lambda \mapsto (U_{h_1} - \lambda U_{h_2})^{-1}(U_{h_3} - \lambda U_{h_4})$ is rational as each component is a rational function, hence the vector bundle is rational as well. Therefore, E_i is a $(d + 1)$-dimensional semialgebraic set and consequently $\Gamma^4_{\substack{a_1,\dots,a_p\\b_1,\dots,b_p}}$ is a semialgebraic set of dimension $(q(n - 1) + d)$. ∎

From Tarski-Seidenberg theorem and Corollary 4.2 of [13] we have that $f_{g_1,\dots,g_{2p}}(\Gamma^4_{\substack{a_1,\dots,a_p\\ b_1,\dots,b_p}})$ is semialgebraic set, of dimension at most $(d+1)$.

Note that $f_{h_1,\dots,h_{2p}}(\Gamma^4_{\substack{a_1,\dots,a_p\\ b_1,\dots,b_p}})$ is homogeneous, i.e. if

$$(x_1,\dots,x_p) \in f_{h_1,\dots,h_{2p}}(\Gamma^4_{\substack{a_1,\dots,a_p\\ b_1,\dots,b_p}})$$

then

$$(\lambda x_1,\dots,\lambda x_p) \in f_{h_1,\dots,h_{2p}}(\Gamma^4_{\substack{a_1,\dots,a_p\\ b_1,\dots,b_p}}),\ \forall \lambda \in \mathbb{R}$$

Thus we conclude that $f_{h_1,\dots,h_{2p}}(\Gamma^4_{\substack{a_1,\dots,a_p\\ b_1,\dots,b_p}}) \cap S^1(\mathscr{V}^p)$ is a semialgebraic set of dimension at most d.

Proposition 3.14. $\Gamma^3_{\substack{a_1,\dots,a_p\\ b_1,\dots,b_p}}$ *is a finite union of linear subspaces of* $\mathscr{V}^2$ *of dimension at most* $p_n = 2d - \gamma_{N-n+1}$.

Proof. From the fact that E_G is a finite set we conclude that, dimension of $\Gamma^3_{\substack{a_1,\dots,a_p\\ b_1,\dots,b_p}}$ is less than

$$\max_{\substack{\boldsymbol{c}\in E_G^{n-1}\\ a,b\in H^*}} \dim\{(x,y)\in \mathscr{V}\times\mathscr{V} : (U_{h_1}-\lambda U_{h_2})x-(U_{h_{2k+1}}-\lambda U_{h_{2k+1}})y = 0,\ \forall k\in[n-1]\}.$$

Notice, however, that whenever $(U_{h_1}-\lambda U_{h_2})x-(U_{h_{2k+1}}-\lambda U_{h_{2k+2}})y = 0$ the vector (x,y) lies inside the kernel $\ker\{u_{h_1}-\lambda U_{h_2} \mid U_{h_{2k+1}}-\lambda U_{h_{2k+2}}\},\ \forall k \in [n-1]$.

Therefore, we get that

$$\begin{aligned}
&\max_{\substack{\lambda\in E_G\\ a,b\in H^*}} \min_{k\in[n-1]} \dim(\ker\{U_{h_1}-\lambda U_{h_2} \mid U_{h_{2k+1}}-\lambda U_{h_{2k+2}}\})\\
&= \max_{\substack{\lambda\in E_G\\ a,b\in H^*}} \min_{k\in[n-1]} \{2d - \operatorname{rank}[U_{h_1}-\lambda U_{h_2} \mid U_{h_{2k+1}}-\lambda U_{h_{2k+2}}]\}\\
&= 2d - \min_{\substack{\lambda\in E_G\\ a,b\in H^*}} \max_{k\in[n-1]} \operatorname{rank}[U_{h_1}-\lambda U_{h_2} \mid U_{h_{2k+1}}-\lambda U_{h_{2k+2}}].
\end{aligned}$$

Next, we make the following two observations:

1. If we chose $h_1 = h_{2k+1}$ and $h_2 = h_{2k+2}$, then

$$\operatorname{rank}[U_{h_1}-\lambda U_{h_2} \mid h_{2k+1}-\lambda U_{h_{2k+1}}] = \operatorname{rank}[U_{h_1}-\lambda U_{h_2}].$$

2. $\operatorname{rank}[U_{h_1} - \lambda U_{h_2}] = \operatorname{rank}[U_{h_1 h_2^{-1}} - \lambda U_{h_2 h_2^{-1}}] = \operatorname{rank}[U_{h_1 h_2^{-1}} - \lambda \mathrm{I}]$.

So, we conclude that

$$\min_{c\in E_G} \max_{k\in[n-1]} \operatorname{rank}[U_{h_1} - \lambda U_{h_2} \mid U_{h_{2k+1}} - \lambda U_{h_{2k+2}}] =$$
$$\min_{\substack{H\subset G \\ |H|=n-1, \mathrm{I}_d \notin H}} \max_{h\in H} \operatorname{rank}[U_h - \lambda \mathrm{I}_d] = \gamma_{N-n+1}$$

Therefore, $\Gamma^3_{\substack{a_1,\ldots,a_p \\ b_1,\ldots,b_p}}$ is a finite union of linear subspaces of dimension at most

$$p_n = 2d - \gamma_{N-n+1}.$$

□

Lemma 3.15. *For fixed $h_1, \ldots, h_{2p} \in G$, a, b the set*

$$f_{h_1,\ldots,h_{2p}}(\Gamma^3_{\substack{a_1,\ldots,a_p \\ b_1,\ldots,b_p}})$$

is a semialgebraic set of dimension at most p_n.

Proof. Recall that the set

$$f_{h_1,\ldots,h_{2p}}(\Gamma^3_{\substack{a_1,\ldots,a_p \\ b_1,\ldots,b_p}})$$

is a finite union of linear subspaces of $\mathscr{V}^p$ of dimension at most $2d - \gamma_{N-n+1} = p_n$. Also because $f_{h_1,\ldots,h_{2p}}(\Gamma^3_{\substack{a_1,\ldots,a_p \\ b_1,\ldots,b_p}}))$ is an open set with respect to topology induced by $f_{h_1,\ldots,h_{2p}}$. We conclude that

$$f_{h_1,\ldots,h_{2p}}(\Gamma^3_{\substack{a_1,\ldots,a_p \\ b_1,\ldots,b_p}})$$

is a semialgebraic set of dimension at most p_n. □

We have shown that $f_{g_1,\ldots,g_{2p}}(\Gamma^3_{\substack{a_1,\ldots,a_p \\ b_1,\ldots,b_p}}))$ is a semialgebraic set of dimension at most p_n. Notice now, that each of these manifolds is homogeneous, i.e. if

$$(x_1, \ldots, x_p) \in f_{g_1,\ldots,g_{2p}}(\Gamma^3_{\substack{a_1,\ldots,a_p \\ b_1,\ldots,b_p}}))$$

then

$$(\lambda x_1, \ldots, \lambda x_p) \in f_{g_1,\ldots,g_{2p}}(\Gamma^3_{\substack{a_1,\ldots,a_p \\ b_1,\ldots,b_p}})), \ \forall \lambda \in \mathbb{R}$$

Thus we conclude that $f_{g_1,\ldots,g_{2p}}(\Gamma^3_{\substack{a_1,\ldots,a_p \\ b_1,\ldots,b_p}})) \cap S^1(\mathscr{V}^p)$ is a semialgebraic set of dimension at most $p_n - 1$.

Proposition 3.16. *For any* $a_i, b_i \in H^*_{m_i}$, $i \in [p]$. $\mathscr{F}^1_{\substack{a_1,\ldots,a_p\\ b_1,\ldots,b_p}}$ *is a nowhere dense set with zero lebesgue measure.*

Proof. Let $B_3 = \Big(f_{g_1,\ldots,g_{2p}}(\Gamma^3_{\substack{a_1,\ldots,a_p\\ b_1,\ldots,b_p}}) \cup f_{g_1,\ldots,g_{2p}}(\Gamma^4_{\substack{a_1,\ldots,a_p\\ b_1,\ldots,b_p}})\Big) \cap S^1(\mathscr{V}^p)$. We showed that B_3 is semialgebraic set of dimension at most $p_n - 1$.

Following the proof of Proposition 3.5 we construct a finite set $\{\mathbf{w}^j\}_{j=1}^M$, a finite cover $\{B_j\}_{j=1}^M$ of B and a map $\mathrm{P}_{2,j} : B_{3,j} \times \mathbb{R}^{p(d-1)} \to \mathscr{V}^p$ by

$$\mathrm{P}_{3,j}(\mathbf{v}, \mathbf{c}) = \sum_{i=1}^{p(d-1)} c_i e_i^{\mathbf{w}^j,\mathbf{v}}.$$

Observe that

$$\bigcup_{j \in M} \mathrm{P}_{3,j}(B_{3,j} \times \mathbb{R}^{p(d-1)}) \supset \mathscr{F}^2_{\substack{a_1,\ldots,a_p\\ b_1,\ldots,b_p}}.$$

Notice that $B_{3,j} \times \mathbb{R}^{p(d-1)}$ is a semialgebraic set of dimension at most $p_n - 1 + p(d-1)$. Because $\mathrm{P}_{3,j}$ is semialgebraic, from theorem 1.12 we conclude that $\mathrm{P}_{3,j}(B_{3,j} \times \mathbb{R}^{p(d-1)})$ is a semialgebraic set of dimension at most $p_n - 1 + p(d-1)$ and because $p \geq p_n \implies p_n - 1 + p(d-1) < pd$ from corollary 1.14 it is a nowhere dense set with zero Lebesgue measure. □

Proof. (Theorem 3.1.) For fixed $p \geq p_n$ and $S \in [p] \times [N]$, recall that the set of p-tuples of vectors $\mathbf{w} = (w_1, \ldots, w_p)$ such that the pair $(\mathbf{w}, S)$ fails to induce an injective embedding $\Phi_{\mathbf{w},S}$ is denoted by $\mathscr{F}_S$.

Recall also that in order to prove that $\mathscr{F}_S$ has zero Lebesgue measure and is nowhere dense, it suffices to show the same for the set $\mathscr{F}_{\substack{a_1,\ldots,a_p\\ b_1,\ldots,b_p}}$ for any $a_i, b_i \in H^*_{m_i}$, $i \in [p]$.

In Chapter 3.1, we showed that $\mathscr{F}_{\substack{a_1,\ldots,a_p\\ b_1,\ldots,b_p}} = \mathscr{F}^1_{\substack{a_1,\ldots,a_p\\ b_1,\ldots,b_p}} \cup \mathscr{F}^2_{\substack{a_1,\ldots,a_p\\ b_1,\ldots,b_p}}$. But if $p \geq p_n$, Proposition 3.16 demonstrates that $\mathscr{F}^1_{\substack{a_1,\ldots,a_p\\ b_1,\ldots,b_p}}$ has zero measure and is nowhere dense, and Proposition 3.10 demonstrates that $\mathscr{F}^2_{\substack{a_1,\ldots,a_p\\ b_1,\ldots,b_p}}$ has zero measure and is nowhere dense. Therefore, Theorem 3.6 is proved.

Remark 3.17. *In Theorem 3.6 we demonstrated that if we use more than one element per Coorbit we need less than* 2d *windows for the construction of an injective embedding. Unfortunately the dimension of the target space can be greater than* 2d *but in [8] we showed that a generic linear projection in* $\mathbb{R}^{2d}$ *preserves both injectivity and stability properties.*

□

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Appendices

A Proof of Proposition 3.4

Proof. Consider the function $r : U_{\mathbf{w}} \to \mathscr{V}^p$ given by

$$r(\mathbf{v}) = (\frac{v_1}{\|v_1\|}, \dots, \frac{v_p}{\|v_p\|}, e_1^{\mathbf{w},\mathbf{v}}, \dots, e_{p(d-1)}^{\mathbf{w},\mathbf{v}}).$$

The first p components are obviously rational functions in $\mathbf{v}$. For the components $m_k(\mathbf{v}) = e_k^{\mathbf{w},\mathbf{v}}$, for $k = 1, \dots, p(d-1)$, we will use induction. Towards this, observe that

$$m_k(\mathbf{v}) = e_k^{\mathbf{w}} - \sum_{j=1}^{p} \frac{1}{\|v_j\|^2} \langle \{0 \oplus v_j \oplus 0\}_p^j, e_k^{\mathbf{w}} \rangle \{0 \oplus v_j \oplus 0\}_p^j - \sum_{j=1}^{k-1} \langle e_j^{\mathbf{w},\mathbf{v}}, e_k^{\mathbf{w}} \rangle e_j^{\mathbf{w},\mathbf{v}}$$

For $k = 1$, $m_1(\mathbf{v}) = \frac{t_1(\mathbf{v})}{\|t_1(\mathbf{v})\|}$, where

$$\begin{aligned} t_1(\mathbf{v}) &= \frac{\langle \{0 \oplus \frac{v_p}{\|v_p\|} \oplus 0\}_p^p, e_1^{\mathbf{w}} \rangle}{\|\{0 \oplus \frac{v_p}{\|v_p\|^2} \oplus 0\}_p^p\|^2} v_1 = \langle \{0 \oplus \frac{v_p}{\|v_p\|} \oplus 0\}_p^p, e_1^{\mathbf{w}} \rangle) v_1 \\ &= \frac{1}{\|v_p\|^2} \langle \{0 \oplus v_p \oplus 0\}_p^p, e_1^{\mathbf{w}} \rangle v_1. \end{aligned}$$

The function $\frac{1}{\|v_p\|}$ is rational in $\mathbf{v}$, and $\langle \{0 \oplus v_p \oplus 0\}_p^p, e_1^{\mathbf{w}} \rangle$ is linear so we conclude that $t_1(\mathbf{v})$ is rational. Now, suppose that $\forall j \in [k]$, m_j is rational.

Then,

$$m_{k+1}(\mathbf{v}) = e^{\mathbf{w}}_{k+1} - \sum_{j=1}^{p} \frac{1}{\|v_j\|^2} \langle \{0 \oplus v_j \oplus 0\}^j_p, e^{\mathbf{w}}_{k+1} \rangle \{0 \oplus v_j \oplus 0\}^j_p - \sum_{j=1}^{k} \langle e^{\mathbf{w},\mathbf{v}}_j, e^{\mathbf{w}}_k \rangle e^{\mathbf{w},\mathbf{v}}_j.$$

We have already shown that the first two terms of $m_{k+1}(\mathbf{v})$ are rational and $\langle e^{\mathbf{w},\mathbf{v}}_j, e^{\mathbf{w}}_k \rangle e^{\mathbf{w},\mathbf{v}}_j$ is rational in $\mathbf{v}$ as product of rational functions. Therefore, for all $k \in [p(d-1)]$, $m_k(\mathbf{v})$ is a rational function. Consequently, $\phi_{\mathbf{w}}$ and $\psi_{\mathbf{w}}$ are rational diffeomorphisms.

Last, we need to show the transition functions are rationals as well. For this, first fix $\mathbf{w}_1, \mathbf{w}_2 \in M$ and $U := U_{\mathbf{w}_1} \cap U_{\mathbf{w}_2} \neq \emptyset$, and let $\sigma_{\mathbf{w}_1,\mathbf{w}_2} : U \times \mathbb{R}^{p(d-1)} \to U \times \mathbb{R}^{p(d-1)}$ be the map $\sigma_{\mathbf{w}_1,\mathbf{w}_2} = \psi_{\mathbf{w}_1} \circ \phi_{\mathbf{w}_2}$. This induces another map $g_{\mathbf{w}_1,\mathbf{w}_2} : U \to GL(p(d-1))$ via $\sigma_{\mathbf{w}_1,\mathbf{w}_2}(\mathbf{v}, c) = (\mathbf{v}, g_{\mathbf{w}_1,\mathbf{w}_2}(\mathbf{v})c)$, where $c = (c_i)_{1 \le i \le p(d-1)}$. It suffices to show that the transition map $g_{\mathbf{w}_1,\mathbf{w}_2}$ is rational. In fact, $g_{\mathbf{w}_1,\mathbf{w}_2}$ is given by

$$g_{\mathbf{w}_1,\mathbf{w}_2}(\mathbf{v})c = \left(\sum_{i=1}^{p(d-1)} \langle e^{\mathbf{w}_2,\mathbf{v}}_i, e^{\mathbf{w}_1,\mathbf{v}}_k \rangle c_i \right)_{1 \le k \le p(d-1)}$$

Note that $g_{\mathbf{w}_1,\mathbf{w}_2}$ represents a change of coordinates between two orthonormal bases, and the Cross-Grammian of

$$\{0 \oplus \frac{v_1}{\|v_1\|} \oplus 0\}^1_p, \{0 \oplus \frac{v_2}{\|v_2\|} \oplus 0\}^2_p \ldots, \{0 \oplus \frac{v_p}{\|v_p\|} \oplus 0\}^p_p, \mathbf{e}^{\mathbf{w}_1,\mathbf{v}}_1, \ldots, \mathbf{e}^{\mathbf{w}_1,\mathbf{v}}_{p(d-1)}$$

and

$$\{0 \oplus \frac{v_1}{\|v_1\|} \oplus 0\}^1_p, \{0 \oplus \frac{v_2}{\|v_2\|} \oplus 0\}^2_p, \ldots, \{0 \oplus \frac{v_p}{\|v_p\|} \oplus 0\}^p_p, \mathbf{e}^{\mathbf{w}_2,\mathbf{v}}_1, \ldots, \mathbf{e}^{\mathbf{w}_2,\mathbf{v}}_{p(d-1)}$$

is a block-diagonal orthogonal matrix. But since the first p components of the two sequences of vectors are the same, we conclude that the Cross-Grammian of $\{\mathbf{e}^{\mathbf{w}_1,\mathbf{v}}_1, \ldots, \mathbf{e}^{\mathbf{w}_1,\mathbf{v}}_{p(d-1)}\}$ and $\{\mathbf{e}^{\mathbf{w}_2,\mathbf{v}}_1, \ldots, \mathbf{e}^{\mathbf{w}_2,\mathbf{v}}_{p(d-1)}\}$ is an orthogonal matrix. But this is exactly $g_{\mathbf{w}_1,\mathbf{w}_2}(\mathbf{v})$ which in turn is rational map in $\mathbf{v}$.

Note that we showed that for fixed i, j the map $e^{\mathbf{w}^i,\mathbf{v}}_j : U_{\mathbf{w}^i} \to \mathscr{V}^p$ is a morphism which is a stronger statement than smoothness or analyticity. □